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Information transfer in generalized probabilistic theories

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ABSTRACT

The tension between unitarity and wave-packet collapse is an annoying problem in quantum mechanics, while a breakthrough was made by Zurek recently from the point of view of information transfer. In this paper, we reconsider Zurek's derivation in the setting of generalized probabilistic theories (GPT), and establish that actionable information about a system can be repeatedly passed on to other systems only when the chosen states of the system have mutual zero fidelity. This may be interpreted as an extension of Zurek's result to GPT.

1. Introduction

It is well known that the indeterminism of measurement outcomes is a prominent symbol in the conventional interpretation of quantum mechanics [1]. Quantum mechanics can be viewed as a class of generalized probabilistic theories (GPT) mathematically. During the past few years, studies on quantum information and quantum computation theory poses an urgent request for a thorough understanding of the foundations of quantum mechanics and how to draw the line between classical and quantum. Among such endeavors, Barnum et al. [2,3] proposed a framework called generalized probabilistic theories which goes beyond either classical or quantum probabilities. Many characteristics and properties previously considered to be possessed only by quantum theory, such as non-locality, monogamy of correlations, uncertainty relations, non-classical secret key distribution, no-broadcasting, state discrimination, Tsirelson's bound, distinguishability measures (e.g., fidelity), entropy, teleportation, etc., are proven to be more general ones, which are revisited and discussed in GPT [3–13].

Motivated by the above observations, it is desirable to consider the issue of unitarity and wave-packet collapse arising from quantum theory in the framework of GPT. The conflict between the two core postulates, unitary evolution versus wave-packet collapse, has been a hard-to-solve problem since the advent of quantum mechanics. Such challenge was overcome in some sense by Zurek, who first derived wave-packet collapse from unitary evolution and repeatability [14]. This creative work turned out to be an extension of quantum no-cloning theorem in the more general setting of information transfer, and has received high evaluation [15]. Illuminated by Zurek's work, Luo presented two new approaches by posing weak repeatability or covariant condition instead of repeatability [16]. Utilizing the properties of the fidelity between states in GPT, Zander and Plastino studied conservation of information during the evolution of a closed physical system in GPT [17], which extends the work of [14]. Wu et al. revisited Luo's derivation in the framework of GPT and generalized the corresponding results [18]. Recently, Zurek studied the scenario of mixed states and concluded that repeatedly accessible states of macroscopic systems must correspond to orthogonal subspaces [19]. In this paper, we extend Zurek's recent work to GPT. Our main result is that information transfer in GPT also induces perfect distinguishability of states.

2. Operational framework of GPT

In this section, we recall the operational framework of GPT (see [3,9,17] for more details):

(1) *State.* The set of all the states of a physical system, called state space and denoted by S, is represented by a compact and convex set of a finite dimensional space. The pure states are extreme points of S. Note that the state space S in quantum me-

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chanics formalism is the set of all density operators ρ on a Hilbert space.

(2) *Effect.* An affine functional $e : S \to [0, 1]$ is called an *effect*, and the probability of e when the system is in state s is $e(s)$. The unit effect ι is an affine functional satisfying $\iota(s) = 1$ for all $s \in S$. The set of all effects is denoted by $\mathcal{E}(S)$. We can embed S in $\mathcal{E}(S)^*$, which maps s to $\hat{s}$ such that $\hat{s}(e) = e(s)$ for all $e \in \mathcal{E}(S)$. So $s(e)$ and $e(s)$ can be seen as equivalent by identifying S with $\hat{S}$. Note that in quantum theory, an effect is an operator E satisfying $0 \le E \le I$, and $\mathrm{Tr}(E\rho)$ is the probability of the effect when the system is in state ρ.

(3) *Measurement.* A set of effects $\{e_i\}$ with $\sum_i e_i = \iota$ is called a measurement, denoted by $M = \{e_i\}$, which implies that $\sum_i e_i(s) = 1$ for all $s \in S$. The set of all measurements is denoted by $\mathcal{M}$.

(4) *Transformation.* A set of affine mappings $T : S \to S'$ represents the physical transformations of a system, where S and S' are both state spaces. Note that in quantum theory, the transformations are described by completely positive trace-preserving maps.

(5) *Composite system.* Consider a composite system AB with subsystems A and B (which are described by state spaces S_A and S_B, respectively). A bi-affine map s_{AB} on $\mathcal{E}(S_A) \times \mathcal{E}(S_B)$ provides a description of a composite state on AB. For a product state $s \otimes t$, it holds that $(s \otimes t)(e, f) = s(e)t(f)$ for states $s \in S_A, t \in S_B$ and effects $e \in \mathcal{E}(S_A)$, $f \in \mathcal{E}(S_B)$. Moreover, $s_A(e) = s_{AB}(e, \iota_B)$ and $s_B(f) = s_{AB}(\iota_A, f)$ defines the marginal states s_A and s_B of $s_{AB} \in S_A \otimes S_B$, respectively, where ι_A and ι_B are the corresponding unit effects on S_A and S_B.

It is worth pointing out that the examples of GPT includes finite classical systems, finite quantum systems, hyper cuboid systems, etc. [9].

3. Information transfer: orthogonality, repeatability, actionable information

In seeking compromise between unitarity and wave-packet collapse, Zurek invoked the postulate of repeatability [14]: After a measurement, repeated measurements leave the system states intact and yield the same results in the apparatus. Consider the paradigm that a quantum system S is measured by an apparatus A. The measurement is conducted by a unitary operator on SA. Based on the above repeatability postulate and the unitarity, Zurek derived the wave-packet collapse from an information transfer perspective. He further addressed the wave-packet collapse problem by considering the mixed state case, and derived the discreteness of quantum jumps from unitarity, repeatability, and actionable information [19], which extended the results in [14] to a more general scenario.

It is pointed out that it seems to be an idealization to assume that the outcome states of the measured system remain the same [16]. Therefore, Luo relaxed the postulate of repeatability by proposing the following one: After a measurement, repeated measurements yield the same results in the apparatus irrespective of the system state is changed or not. He also further considered another postulate which depends on a unitary covariant condition for the apparatus without resorting to any repeatability.

Note that in either derivation of the above-mentioned two papers, the measurement is carried out by a unitary operator on the composite system SA. In this work, we shall use invertible transformations as more general ones. In fact, in GPT, it is assumed that a closed physical system with a state space S will evolve under the action of invertible transformations $\Gamma : S \to S$, which play the role of unitaries in quantum mechanics. Such a transformation Γ has an inverse Γ^{-1} satisfying that $\Gamma^{-1}(\Gamma(s)) = \Gamma(\Gamma^{-1}(s)) = s$ for each $s \in S$, which can be regarded as a natural generalization of $U^\dagger U = UU^\dagger = I$ in quantum theory. Any transformation on a system can be realized by an invertible transformation acting upon a certain extended closed system [17], just like the action of a quantum operation on a quantum state can be regarded as a reduction of a unitary evolution of the corresponding tensor product state in the extended closed quantum system. This assumption is satisfied by both classical and quantum mechanics, and will be adopted in our discussion in GPT.

In order to give our general derivation, we first recall the definition of the fidelity between two states $s_1, s_2 \in S$ in GPT [9,17]:

$$F(s_1, s_2) = \inf_M F_c(p_1(M), p_2(M)), \tag{1}$$

where inf is over all measurements $M = \{e_i\}$, $p_1(M) = \{e_i(s_1)\}$ and $p_2(M) = \{e_i(s_2)\}$, and $F_c(x, y) = \sum_i \sqrt{x_i y_i}$ is the classical fidelity between two probability distributions $x = \{x_i\}$ and $y = \{y_i\}$. Two states in GPT are defined as orthogonal if the fidelity between them is zero. This is apparently the direct generalization of orthogonality of pure states in quantum mechanics. The following lemma will play an important role in our derivation [9,17].

Lemma 1. *Suppose that S_A and S_B are two state spaces in GPT, then*

(i) $F(s_1 \otimes t_1, s_2 \otimes t_2) \le F(s_1, s_2)F(t_1, t_2)$, $\forall\, s_1, s_2 \in S_A, t_1, t_2 \in S_B$;
(ii) $F(s_1, s_2) = F(s_1 \otimes t, s_2 \otimes t)$, $\forall\, s_1, s_2 \in S_A, t \in S_B$.

The notion of measurement is primitive in GPT. Measurements and transformations in GPT are closely related to each other. On the one hand, let $M = \{e_i\} \in \mathcal{M}$ be a measurement on a state space S, and $\{s'_i\}$ be a set of states in another state space S', then it is easy to check that the mapping $\phi : S \to S'$ given by $\phi(s) = \sum_i e_i(s)s'_i$ is affine, i.e., ϕ is a transformation which is induced by a measurement [2]. On the other hand, conducting an invertible transformation Γ and a measurement $M = \{e_i\}$ sequentially on a system is equivalent to posing another measurement $\tilde{M} = \{\tilde{e}_i\}$ on the system. In other words, for all states $s \in S$, it holds that $\tilde{e}_i(s) = e_i(\Gamma(s))$. From this observation, it is not difficult to deduce the following equation by using Eq. (1) [17]:

$$F(\Gamma(s_1), \Gamma(s_2)) = F(s_1, s_2), \tag{2}$$

where Γ is any invertible transformation. This property extends the invariance property of the quantum fidelity under unitary evolutions in quantum theory, and will be used in the following derivations in GPT.

We are now ready to extend Zurek's derivation to the framework of GPT. Consider a system S starting from a mixed state s^u, and a set of measurement apparatuses, $A_1, A_2, \ldots, A_k, \ldots$, with initial pure states, $a_1, a_2, \ldots, a_k, \ldots$. Then under a sequence of information transfer realized by invertible transformations, $\Gamma_1, \Gamma_2, \ldots, \Gamma_k, \ldots$, we get

$$\begin{aligned} &s^u \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_k \xrightarrow{\Gamma_1} s_1^u \otimes a_1^u \otimes a_2 \otimes \cdots \otimes a_k \\ &\xrightarrow{\Gamma_2} s_2^u \otimes a_1^u \otimes a_2^u \otimes \cdots \otimes a_k \xrightarrow{\Gamma_3} \cdots. \end{aligned} \tag{3}$$

The above procedure shows that the information about the state identity u is first transferred to the measurement apparatus A_1 (whose state suffers the change $a_1 \to a_1^u$ and records the system state information u), and then is transferred to A_2, etc. This is just a manifestation of the repeatability postulate. Now what we care about is: Under the above scenario of information transfer, if distinguishable copies of the system state information exists in the measurement apparatuses, what features the set of initial mixed states of the system S should have? To this end, we assume that another system state s^v goes through the exactly same transfer procedure as Eq. (3):