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Entangled entanglement: A construction procedure

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ABSTRACT

The familiar Greenberger–Horne–Zeilinger (GHZ) states can be rewritten by entangling the Bell states for two qubits with a third qubit state, which is dubbed *entangled entanglement*. We show that in a constructive way we obtain all eight independent GHZ states that form the simplex of entangled entanglement, the *magic simplex*. The construction procedure allows a generalization to higher dimensions both, in the degrees of freedom (considering qudits) as well as in the number of particles (considering n-partite states). Such bases of GHZ-type states exhibit a cyclic geometry, a *Merry Go Round*, that is relevant for experimental and quantum information theoretic applications.

1. Introduction

Entanglement as one of the most fundamental phenomena in quantum physics has many fascinating aspects. An amazing feature that occurs for multipartite systems is *entangled entanglement*. The term was coined by Krenn and Zeilinger [1] to characterize the phenomenon that the entanglement of two qubits, expressed by the Bell states, can be entangled further with a third qubit, producing such a particular Greenberger–Horne–Zeilinger (GHZ) state. We take up this idea, develop it further and show that all independent (maximally entangled) GHZ states, can be expressed, geometrically quite obviously, in an entangled entanglement form. These basis states configure the *magic simplex* [2]. The word "magic" goes back to Wootter's magic basis to compute the concurrence [3]. We show then explicitly how our construction procedure, which is entirely systematic and intuitive, can be generalized to higher dimensions d and to any finite number of particles n, namely to n-partite qudit states $d \otimes d \otimes d \otimes d \otimes \ldots \otimes d = d^{\otimes n}$.

To obtain an understanding and intuition of the physics behind entangled entanglement we discuss the case of the tripartite GHZ states in Section 2, on one hand, with respect to the Einstein–Podolsky–Rosen (EPR) paradox, and on the other hand, with reference to the mathematical structure, the freedom to factorize a tensor product of algebras (or Hilbert spaces) in different ways, which forms the mathematical basis for the phenomenon of entangled entanglement. In Section 3 we introduce our procedure how to construct systematically the states of entangled entanglement for any higher dimension and number of particles. The use of the unitary Weyl operators [4] turns out to be very helpful (also known under names like "generalized spin operators", "Pauli group" and "Heisenberg group", Refs. [5–8]). In Weyl's book these unitary operators, consisting of phase and (cyclic) ladder operators, were introduced by a "quantization" of classical kinematics that is the reason why the magic simplex is sometimes considered as a phase-space.

We also illustrate the geometric structure of the state space (see Fig. 2), the symmetries inherent in a magic simplex and the cyclicity of the phase operations, when moving from one state to another within the simplex. In particular we discover a *Merry Go Round* of the qutrit GHZ states (see Fig. 3). Finally, conclusions are drawn in Section 4.

2. Physical aspect and mathematical structure

Let us begin with discussing the physics behind the phenomenon of entangled entanglement. We recall the well-known GHZ state [9,10]

$$|\mathrm{GHZ1}^-\rangle_{123} = \frac{1}{\sqrt{2}} \left(|R\rangle_1 \otimes |R\rangle_2 \otimes |R\rangle_3 +, |L\rangle_1 \otimes |L\rangle_2 \otimes |L\rangle_3 \right) , \tag{1}$$

where $|R\rangle, |L\rangle$ denote the right- and left-handed circularly polarized photons. Interestingly, expression (1) can be re-expressed by decomposing (1) into linearly polarized states $|H\rangle, |V\rangle$ and Bell states

$$|\mathrm{GHZ1}^-\rangle_{123} = \frac{1}{\sqrt{2}} \left(|H\rangle_1 \otimes |\phi^-\rangle_{23} - |V\rangle_1 \otimes |\psi^+\rangle_{23} \right) , \tag{2}$$

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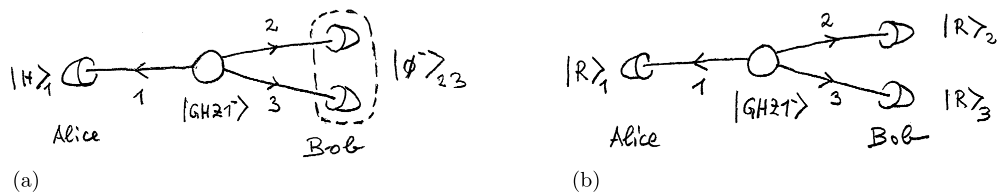

Fig. 1. (a) Bob's photons are in an entangled state. (b) Bob's photons are in a separable state.

where $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle\otimes|H\rangle \pm |V\rangle\otimes|V\rangle\right)$, $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle\otimes|V\rangle \pm |V\rangle\otimes|H\rangle\right)$ represent the familiar maximally entangled Bell states. The linearly polarized states $|H/V\rangle$ are related to the circularly polarized states via $|R/L\rangle = \frac{1}{\sqrt{2}}(|H\rangle \pm i|V\rangle)$.

The GHZ state as expressed in Eq. (2) obviously represents entangled entanglement. This feature has been verified experimentally by Zeilinger's group [11] who has performed a Bell-type experiment on three particles, where one part, Alice on line 1, projects onto the horizontally $|H\rangle_1$ or vertically $|V\rangle_1$ polarized state and the other part, Bob on lines 2 and 3, projects onto the maximally entangled states $|\phi^-\rangle_{23}$ or $|\psi^+\rangle_{23}$ via a Bell state measurement based on a polarizing beam splitter [12]. Then the authors test a Clauser–Horne–Shimony–Holt inequality established between Alice and Bob and find a strong violation of the inequality (specifically, of the Bell parameter) by more than five standard deviations. Thus the entangled states of the two photons on Bob's side are definitely entangled again with the single photon on Alice's side.

What is the physical significance of it, in particular, in the light of an EPR reasoning? Let us start with an EPR-like discussion as in Ref. [11]. If Alice is measuring the linearly polarized state $|H\rangle_1$ then Bob will find the Bell state $|\phi^-\rangle_{23}$ for his two photons (see Fig. 1(a)). If she obtains a $|V\rangle_1$ state in her measurement then Bob will get the Bell state $|\psi^+\rangle_{23}$. This perfect correlation between the polarization state of one photon on Alice's side and the entangled state of the two photons on Bob's side implies, under the EPR premises of realism and "no action at a distance", that the entangled state of the two photons must represent an element of reality. Whereas the individual photons of this state, which have no well-defined property, do not correspond to such elements. For a realist this is a surprising feature, indeed.

If, on the other hand, Alice is measuring a right-handed circularly polarized state $|R\rangle_1$ then Bob will find his two photons in a separable state $|R\rangle_2 \otimes |R\rangle_3$ (see Fig. 1(b)), or if Alice measures $|L\rangle_1$ Bob will get $|L\rangle_2 \otimes |L\rangle_3$. Then the two photons of Bob contain individually an element of reality, which is more satisfactory to a realist. Thus by the specific kind of measurement, projecting on linearly or circularly polarized photons, Alice is able to switch on Bob's side the properties of the two photons — and their reality content — between entanglement and separability.

This feature is even more puzzling in case of entangling internal with external degrees of freedom, which is experimentally achieved in neutron interferometry. The experimenters of Ref. [13] produced a GHZ-like state for single neutrons entangled in path–spin–energy. There the above considerations also have to hold.

How can we understand this switching phenomenon between entanglement and separability? A quantum theorist can trace this switch back to two different factorizations of the tensor product of three algebras $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$, where $\mathcal{A}_1$ belongs to Alice and $\mathcal{A}_2 \otimes \mathcal{A}_3$ to Bob. There is total democracy between the different factorizations [14,15], no partition has ontologically a superior status over any other one (if no specific physical realization is taken into account). For an experimentalist, however, a certain factorization is preferred and is clearly fixed by the set-up.

For tripartite states, the GHZ states, which are defined on a tensor product of three algebras, there exists the following theorem [14], where $\rho = |\psi\rangle\langle\psi|$ denotes the corresponding density matrix of the quantum state $|\psi\rangle$:

Theorem 1 *(Factorization algebra). For any pure tripartite state ρ one can find a factorization $M = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$ such that ρ is separable with respect to this factorization and another factorization $M = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_3$ where ρ appears to be maximally entangled.*

For mixed states, however, such a unitary switching between separable and entangled states exists only beyond a certain bound of mixedness [14].

Example. To illustrate Theorem 1 we consider the circularly polarized states $\{|R\rangle, |L\rangle\}$. We find, for example, the following unitary matrix $U^\dagger$ that transforms the separable state $|R\rangle_1 \otimes |R\rangle_2 \otimes |R\rangle_3$ into the entangled state $|\mathrm{GHZ1}^-\rangle_{123}$ of Eq. (1)

$$U^\dagger |R\rangle_1 \otimes |R\rangle_2 \otimes |R\rangle_3 = |\mathrm{GHZ1}^-\rangle_{123}\,, \tag{3}$$

where

$$U = U_0^{\mathrm{T}\otimes 3} \cdot U_{\mathrm{ent}} \cdot U_0^{\otimes 3} \qquad \text{with} \qquad U_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tag{4}$$

and

$$U_{\mathrm{ent}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{5}$$

Having found the structure of entangled entanglement, it is quite natural to ask if other GHZ states can be expressed in a similar way. The answer is yes, we can construct a complete orthonormal system. Geometrically it is quite obvious how to proceed. We just have to entangle the opposite states $|\phi^-\rangle$ and $|\psi^+\rangle$ or $|\phi^+\rangle$ and $|\psi^-\rangle$ of the $2\otimes 2$ dimensional tetrahedron of Bell states [16–18] with $|H\rangle$ and $|V\rangle$, and respect the symmetric and antisymmetric property respectively. In this way we immediately find an orthonormal basis of eight states

$$\begin{aligned}
|\mathrm{GHZ1}^+\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|H\rangle_1 \otimes |\phi^-\rangle_{23} + |V\rangle_1 \otimes |\psi^+\rangle_{23}\right)\\
|\mathrm{GHZ1}^-\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|H\rangle_1 \otimes |\phi^-\rangle_{23} - |V\rangle_1 \otimes |\psi^+\rangle_{23}\right)\\
|\mathrm{GHZ2}^+\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|H\rangle_1 \otimes |\phi^+\rangle_{23} + |V\rangle_1 \otimes |\psi^-\rangle_{23}\right)\\
|\mathrm{GHZ2}^-\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|H\rangle_1 \otimes |\phi^+\rangle_{23} - |V\rangle_1 \otimes |\psi^-\rangle_{23}\right)\\
|\mathrm{GHZ3}^+\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|V\rangle_1 \otimes |\phi^-\rangle_{23} + |H\rangle_1 \otimes |\psi^+\rangle_{23}\right)\\
|\mathrm{GHZ3}^-\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|V\rangle_1 \otimes |\phi^-\rangle_{23} - |H\rangle_1 \otimes |\psi^+\rangle_{23}\right)\\
|\mathrm{GHZ4}^+\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|V\rangle_1 \otimes |\phi^+\rangle_{23} + |H\rangle_1 \otimes |\psi^-\rangle_{23}\right)\\
|\mathrm{GHZ4}^-\rangle_{123} &= \tfrac{1}{\sqrt{2}}\left(|V\rangle_1 \otimes |\phi^+\rangle_{23} - |H\rangle_1 \otimes |\psi^-\rangle_{23}\right).
\end{aligned} \tag{6}$$