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Bound states in the continuum generated by supersymmetric quantum mechanics and phase rigidity of the corresponding wavefunctions

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ABSTRACT

Supersymmetric quantum mechanics (SUSYQM) is a method that can be used for generating complex potentials with entirely real spectrum with bound states in the continuum (BIC). These complex potentials are isospectral with the initial one, but SUSYQM method adds discrete BIC's at selected energies. Corresponding wavefunctions created by SUSYQM are biorthogonal and complex, hence we can discuss their phase rigidity and illustrate the application of SUSYQM on the examples of three specific potential profiles (free electron, negative Dirac potential and quantum well with infinite walls).

1. Introduction

Bound states in the continuum were first considered by Von Neumann and Wigner [1] who modulated the wavefunction in order to make it normalizable, and then used the modulating function to extract the potential which supports such states. Herrick and Stillinger [2–4] showed that the bound states in the continuum may exist in atoms and molecules, and also pointed to the possibility of an electron in electric field becoming localized by adding a suitable potential. Starting with a separable form of the Hamiltonian, Robnik also derived normalizable wavefunctions [5]. Various techniques have been employed for the wavefunction modulation [6]. Concerning the localization problem, a tight relationship has been established between the asymptotic behavior of the potential envelope and the wavefunction [7].

Supersymmetric quantum mechanics (SUSYQM) is a method that can be used to obtain bound states in the continuum part of the spectrum. In order to get bound states on the full line we introduce a complex potential (applying this technique to a real potential leads to bound states only on the half-line $x \in (0, +\infty)$ [8,9]) isospectral with the initial one. Each of the complex potentials supports one and only one normalizable wavefunction in the continuum part of the spectrum. Real spectra with complex potentials also discussed in [10–15]. In [11–13] focus was on broadening a condition of hermiticity to partial reflectivity and time reversal symmetry of Hamiltonian and in [14] authors linked some of these Hamiltonians to SUSY transformation, in [15] pseudo-hermiticity was introduced in order to generalize real spectra with complex potentials even further.

Since the form of the Schrödinger's equation is similar to the Helmholtz equation (with the term labeled 'the optical potential'), it is possible to apply SUSYQM and study BIC's in photonic crystals [16–18], one dimensional heterostructures [19] and quasiperiodic systems [20].

The wavefunctions created by SUSYQM [21,22] are complex as well, so we can consider their phase rigidity. Phase rigidity represents the measure of the complexity of the wavefunction, if its value equals 1 wavefunction is real and if it equals 0, wavefunction is fully complex. It can be physically represented as the measure of the alignment between one of the overlapping resonance states with one of the scattering states of the environment in open quantum systems [23,24]. Phase rigidity has also been used to characterize mode–mode interaction in dielectric microcavities [25], linewidth of lasing modes in unstable lasers [26,27], excess spontaneous emission in open sided laser resonators [28], tunability of passage time in realistic open quantum systems due to biorthogonality of the eigenfunction of non-Hermitian Hamiltonian [29], one-dimensional point interactions [30] and transmission through a quantum dot [31].

In [21] authors discussed the normalizability of these wavefunctions depending on constants C and λ. In both papers [21,22] C was considered to be strictly complex, and λ to be real. For standard SUSYQM $C = 0$ and λ is real. In this paper we will focus on case where $C = 0$ and λ is strictly complex, because this case is not covered in [21,22] and the main results of SUSYQM will be significantly different than those presented in [21,22]. We perform

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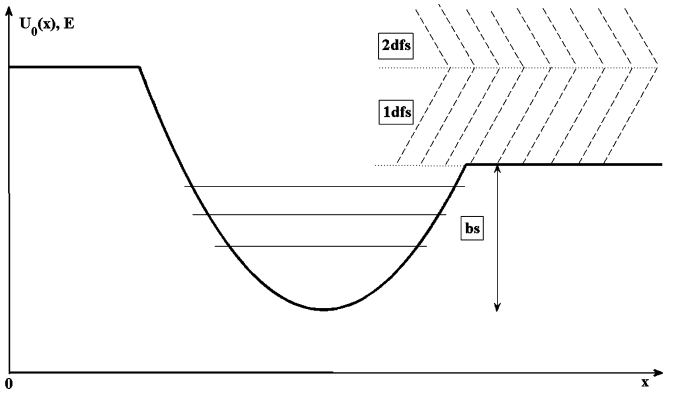

Fig. 1. The real potential $U_0(x)$, which supports bound states, non-degenerate free states (1dfs), and double-degenerate free states (2dfs). Energy $\varepsilon > \min(U_0)$ in the main text, represents any energy from the spectrum of $U_0(x)$.

an analysis of generated wavefunctions depending on the type of spectrum that a chosen state belongs to (in the initial potential) and prove that the chosen state will be a bound state (in the complex potential). Considering λ to be complex will yield constrains for normalizability, which will be discussed.

2. Theoretical consideration

We consider a general real potential $U_0(x)$ which supports continuum (free) and bound states (Fig. 1). After solving Schrödinger's equation we choose an arbitrary energy $\varepsilon > \min(U_0)$, its corresponding wavefunction denoted as Ψ_ε. For all other energies E_i, except the chosen energy ε ($E_i \neq \varepsilon$) corresponding wavefunctions are labeled as Ψ_i. Both Ψ_ε and Ψ_i do not need to be normalized. Without loss of generality, we shall consider Ψ_ε and Ψ_i as real functions.

We apply the SUSYQM procedure via energy ε, which results in a potential:

$$U_{ss}(x) = U_0(x) - 2\alpha^2 \frac{d^2}{dx^2} \ln\big(\lambda + I(x)\big), \quad \alpha^2 = \frac{\hbar^2}{2m} \tag{1}$$

where $I(x)$ is:

$$I(x) = \int_{x_0}^{x} |\Psi_\varepsilon|^2 dx = \int_{x_0}^{x} \Psi_\varepsilon^2 dx. \tag{2}$$

In the above expression, λ has to be considered as a strictly complex constant and it can be represented as $\lambda = \lambda_r + i\lambda_i$, where λ_r, λ_i and x_0 are real parameters. $\Psi_{ss\varepsilon}(x)$ is a wavefunction in the complex potential U_{ss} which corresponds to energy ε:

$$\Psi_{ss\varepsilon}(x) = \frac{\Psi_\varepsilon(x)}{\lambda + I(x)}. \tag{3}$$

Furthermore, $\Psi_{ssi}(x)$ are the wavefunctions in the new potential U_{ss} which correspond to energies $E_i \neq \varepsilon$:

$$\Psi_{ssi}(x) = \Psi_i(x) - \frac{\Psi_\varepsilon \mu_i}{(\lambda + I(x))},$$

$$\mu_i = \int_{-\infty}^{x} \Psi_i \Psi_\varepsilon dx. \tag{4}$$

The functions $\Psi_{ss\varepsilon}(x)$ and $\Psi_{ssi}(x)$ are complex and non-normalized.

Derivation of (1)–(4) can be found in Appendix A.

Table 1
Asymptotic values of integral I depending on energy ε.

$x \to -\infty$	$x \to \infty$	ε
0	$r = const$	bound state
0	$\frac{x}{2}$	non-degenerated free state
$\frac{x}{2}$	$\frac{x}{2}$	degenerated free state

Table 2
Integral J depending on energy ε.

Integral J	ε
$\frac{1}{\lvert\lambda_i\rvert}(\operatorname{atan}\frac{r+\lambda_r}{\lvert\lambda_i\rvert} - \operatorname{atan}\frac{\lambda_r}{\lvert\lambda_i\rvert})$, $\lambda_i \neq 0$	bound state
$\frac{1}{\lvert\lambda_i\rvert}(\frac{\pi}{2} - \operatorname{atan}\frac{\lambda_r}{\lvert\lambda_i\rvert})$, $\lambda_i \neq 0$	non-degenerated free state
$\frac{\pi}{\lvert\lambda_i\rvert}$, $\lambda_i \neq 0$	degenerated free state

2.1. Normalization of $\Psi_{ss\varepsilon}(x)$

Let us consider the integral:

$$J = \int_{-\infty}^{\infty} \big|\Psi_{ss\varepsilon}(x)\big|^2 dx = \int_{-\infty}^{\infty} \frac{\Psi_\varepsilon^2}{|\lambda + I(x)|^2} dx. \tag{5}$$

By differentiating (2) to obtain $dI = \Psi_\varepsilon^2 dx$, and inserting it into Eq. (5) we can calculate J in the form:

$$J = \frac{1}{|\lambda_i|} \operatorname{atan}\left(\frac{I(x) + \lambda_r}{|\lambda_i|}\right)\Bigg|_{I(-\infty)}^{I(+\infty)}. \tag{6}$$

The normalized wavefunction $\Psi_{ss\varepsilon}^N(x)$ then reads $\Psi_{ss\varepsilon}^N(x) = J^{-\frac{1}{2}} \times \Psi_{ss\varepsilon}(x)$. This applies only if J is finite and it is required that $\Psi_{ss\varepsilon}(x)$ has no singularities. Depending on the part of the spectrum of $U_0(x)$ that energy ε belongs to, we get different results for J and the corresponding wavefunction.

Table 1 represents asymptotic values of integral I (defined in (2)) and it is the consequence of properties of wavefunction $\Psi_\varepsilon(x)$. Parameter x_0 is chosen so to satisfy wavefunction normalizability, for non-degenerated free states and bound states is taken $x_0 = -\infty$ and for degenerated free states it can be any finite value and we can choose $x_0 = 0$, but generally results of Table 1 do depend on this parameter. In all SUSYQM results (equations (1), (2) and (4)) integral $I(x)$ is followed by parameter λ, thus for any type of state in the spectrum all dependence on x_0 on results can be joined into parameter λ_r and x_0 wouldn't influence final results. By applying properties from Table 1 to (5) we can calculate integral J. The results are shown in Table 2.

It is necessary for wavefunction $\Psi_{ss\varepsilon}^N$ not to have singularities hence integral J has to be finite value thus we can further discuss results from Table 2 depending on parameter λ.

If the energy ε represents a bound state ($\varepsilon \in$ bs) of the potential $U_0(x)$ then if $\lambda_i \to 0$ and $\lambda_r \notin [-r, 0]$:

$$J \to \frac{r}{\lambda_r(r + \lambda_r)}. \tag{7}$$

In case when $\lambda_i \to 0$, $\Psi_{ss\varepsilon}^N(x)$ has a singularity for $\lambda_r \in [-r, 0]$, which is illustrated in Fig. 2.

If the energy ε represents a non-degenerate free state ($\varepsilon \in$ 1dfs) of the potential $U_0(x)$ then if $\lambda_i \to 0$:

$$\lim_{\lambda_i \to 0}(J) = \begin{cases} \frac{1}{\lambda_r}, & \lambda_r \geq 0 \\ \frac{\pi}{|\lambda_i|} \to +\infty, & \lambda_r < 0. \end{cases} \tag{8}$$