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The Gaussian radial basis function method for plasma kinetic theory

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ABSTRACT

Description of a magnetized plasma involves the Vlasov equation supplemented with the non-linear Fokker–Planck collision operator. For non-Maxwellian distributions, the collision operator, however, is difficult to compute. In this Letter, we introduce Gaussian Radial Basis Functions (RBFs) to discretize the velocity space of the entire kinetic system, and give the corresponding analytical expressions for the Vlasov and collision operator. Outlining the general theory, we also highlight the connection to plasma fluid theories, and give 2D and 3D numerical solutions of the non-linear Fokker–Planck equation. Applications are anticipated in both astrophysical and laboratory plasmas.

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1. Motivation

A fundamental macroscopic description of a magnetized plasma is the Vlasov equation supplemented by the non-linear inverse-square force Fokker–Planck collision operator [1]

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\nabla f_a + \frac{e_a}{m_a}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C_{ab}(f_a, f_b)\,, \qquad (1)$$

where f_a is the distribution of species a with charge e_a and mass m_a. The electric and magnetic fields depend on the distribution f_a through Maxwell's equations. This model assumes a statistical description of Coulomb interaction in the limit of small-angle scattering, with the changes in f_a due to collisions with species b described by

$$C_{ab} = \frac{\partial}{\partial \mathbf{v}}\cdot\left[\mathbf{A}_{ab} f_a + \frac{\partial}{\partial \mathbf{v}}\cdot(\mathbb{D}_{ab} f_a)\right].$$

The friction and diffusion coefficients are given by the expressions

$$\mathbf{A}_{ab} = L_{ab}\left(1+\frac{m_a}{m_b}\right)\frac{\partial \varphi_b}{\partial \mathbf{v}}\,, \qquad \mathbb{D}_{ab} = -L^{ab}\frac{\partial^2 \psi_b}{\partial \mathbf{v}\partial \mathbf{v}}\,,$$

where $L_{ab} = (e_a e_b/m_a\varepsilon_0)^2 \ln\Lambda_{ab}$ and $\ln\Lambda_{ab}$ is the Coulomb logarithm which represents a physical cut-off for small-angle collisions. The Rosenbluth potentials appearing in the friction and diffusion coefficients are weighted integrals of the distribution function

$$\varphi_b(\mathbf{x},\mathbf{v},t) = -\frac{1}{4\pi}\int d\mathbf{v}'\, f_b(\mathbf{x},\mathbf{v}',t)\frac{1}{|\mathbf{v}-\mathbf{v}'|}$$

$$\psi_b(\mathbf{x},\mathbf{v},t) = -\frac{1}{8\pi}\int d\mathbf{v}'\, f_b(\mathbf{x},\mathbf{v}',t)|\mathbf{v}-\mathbf{v}'|$$

and they satisfy the velocity-space Poisson equations $\nabla_{\mathbf{v}}^2\psi_b = \varphi_b$ and $\nabla_{\mathbf{v}}^2\varphi_b = f_b$. Expressed in terms of the potential functions, the Fokker–Planck collision operator is

$$C_{ab} = L_{ab}\left[\frac{m_a}{m_b} f_a f_b + \mu_{ab}\frac{\partial \varphi_b}{\partial \mathbf{v}}\cdot\frac{\partial f_a}{\partial \mathbf{v}} - \frac{\partial^2 \psi_b}{\partial \mathbf{v}\partial \mathbf{v}} : \frac{\partial^2 f_a}{\partial \mathbf{v}\partial \mathbf{v}}\right],$$

where $\mu_{ab} = m_a/m_b - 1$. A common approach for numerical evaluation of C_{ab} follows a two-phase method where one first inverts the velocity-space Laplacian operators and then directly evaluates the collision operator. Boundary conditions for the Poisson equations can be obtained by limiting the solution to a sub-space and evaluating the expressions at the boundary using a multipole expansion of the potentials [2], or by imposing the free-space solution outside the sub-space [3]. Another sophisticated approach is based on fast spectral decomposition as described in Ref. [4].

In this Letter, we propose a new approach using a mesh-free shifted-Maxwellian representation which is intuitively appealing and straightforward to implement. The solution thus obtained is C^∞ smooth, extends to $v\to\infty$, and allows compact representation of any interesting macroscopic quantity (number, momentum, energy density, and so on). The Letter is organized as follows: In Section 2 we present the idea for using Gaussian radial basis functions

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to solve the kinetic equation. The numerical implementation is described in Section 3 and simulations in 2D and 3D velocity space are presented in Section 4. Finally, we discuss and summarize our results in Sections 5 and 6.

2. The Gaussian RBF method

To solve the kinetic equation, Eq. (1), we write the total distribution as a finite sum of shifted Maxwellians

$$f_a(\mathbf{x},\mathbf{v},t)=\sum_i w_a^i(\mathbf{x},t)\,F_a^i(\mathbf{x},\mathbf{v})\,,$$

where each Maxwellian, $F_a^i=\left(\gamma_a^i/\pi\right)^{3/2}\exp\left[-\gamma_a^i(\mathbf{v}-\mathbf{v}_a^i)^2\right]$, is normalized to unity and the weights w_a^i allowed to evolve in time. The width parameters γ_a^i and mean velocities $\mathbf{v}_a^i$ can be arbitrary functions of position. The shifted Maxwellians are nothing other than Gaussian *Radial Basis functions* (RBFs) which have found numerous applications in applied mathematics – in particular for the construction of neural networks [5]. For compactness, in what follows we will retain the spatial dependence of all quantities but will not write the dependence explicitly.

The potential functions then take the form

$$\varphi_a(\mathbf{v},t)=\sum_i w_a^i(t)\,\varphi_a^i(\mathbf{v})\,,\quad \psi_a(\mathbf{v},t)=\sum_i w_a^i(t)\,\psi_a^i(\mathbf{v}),$$

where the Gaussian RBF potentials $\varphi_a^i=-\sqrt{\gamma_a^i}\Phi(\sqrt{\gamma_a^i}|\mathbf{v}-\mathbf{v}_a^i|)/(4\pi)$ and $\psi_a^i=-(1/\sqrt{\gamma_a^i})\Psi(\sqrt{\gamma_a^i}|\mathbf{v}-\mathbf{v}_a^i|)/(8\pi)$ are defined in terms of the functions $\Phi(s)=\mathrm{erf}(s)/s$ and $\Psi(s)=[s+1/(2s)]\mathrm{erf}(s)+\exp(-s^2)/\sqrt{\pi}$, where $\mathrm{erf}(s)$ is the error function. We thus find a simple bilinear expression for the complete non-linear operator

$$C_{ab}=\sum_{k,\ell} w_a^k(t)\,w_b^\ell(t)\,C_{ab}^{k\ell}(\mathbf{v})\,,$$

where the Gaussian RBF collision tensor is

$$C_{ab}^{k\ell}=L_{ab}\left[\frac{m_a}{m_b}F_a^kF_b^\ell+\mu_{ab}\frac{\partial\varphi_a^k}{\partial\mathbf{v}}\cdot\frac{\partial F_b^\ell}{\partial\mathbf{v}}-\frac{\partial^2\psi_a^k}{\partial\mathbf{v}\partial\mathbf{v}}:\frac{\partial^2F_b^\ell}{\partial\mathbf{v}\partial\mathbf{v}}\right],$$

such that $C_{aa}^{kk}(\mathbf{v})=0$. As we have analytical expressions for $F_a^i(\mathbf{v})$, $\varphi_a^i(\mathbf{v})$, and $\psi_a^i(\mathbf{v})$, the tensor $C_{ab}^{k\ell}(\mathbf{v})$ is easy to implement and fast to evaluate at any point in velocity space.

In problems with azimuthal symmetry, a 2D RBF scheme can be developed with axisymmetric *ring*-like RBF-basis:

$$F_a^i=(\gamma_a^i/\pi)^{3/2}I_0(2\gamma_a^iv_{a,\perp}^iv_\perp)\,e^{-\gamma_a^i(v_\parallel-v_{a,\parallel}^i)^2-\gamma_a^i(v_\perp^2+(v_{a,\perp}^i)^2)}\,,$$

where I_0 is the order-zero modified Bessel function of the first kind and $(v_\perp,v_\parallel)$ are the cylindrical velocity-space coordinates. Although explicit expressions for axisymmetric RBF potentials φ_i and ψ_i are not available in a closed form, they can be evaluated numerically to machine precision at any requested point.

3. Collocation options

We describe two different methods for obtaining an ordinary differential equation for the time evolution of weights: the *Galerkin* and the *center-collocation* projections. In the Galerkin method, the kinetic equation – already expanded in the RBF basis – is multiplied by each individual basis function and then integrated over the entire domain. Conversely, in the center-collocation method, the kinetic equation is evaluated at the center of each RBF. Both methods yield N equations for the N RBF weights, w_a^i, and result in a differential equation for the weights that can be written in a matrix form.

For the moment, let us illustrate the method by considering the spatially homogeneous case with no electromagnetic fields. Then, the matrix equation is

$$\sum_j M_a^{ij}\frac{\partial w_a^j}{\partial t}=\sum_{b,k,\ell} w_a^k(t)\,w_b^\ell(t)\,C_{ab}^{ik\ell}\quad\forall\quad i\in 1,2,3,\ldots. \tag{2}$$

In the Galerkin projection (GP), the matrix M_{ij} is symmetric, typically diagonally dominant, and given by the expression

$$(M_a^{ij})_{\mathrm{GP}}=\left(\frac{\gamma_a^i\gamma_a^j}{\pi(\gamma_a^i+\gamma_a^j)}\right)^{3/2}\exp\left[-\frac{\gamma_a^i\gamma_a^j}{\gamma_a^i+\gamma_a^j}\left(\mathbf{v}_a^i-\mathbf{v}_a^j\right)^2\right],$$

whereas in the center-collocation (CC) method, the matrix M_a^{ij} is no longer necessarily symmetric, but can still be dominated by the diagonal components:

$$(M_a^{ij})_{\mathrm{CC}}\equiv F_a^j(\mathbf{v}_a^i)=(\gamma_a^j/\pi)^{3/2}\exp[-\gamma_a^j(\mathbf{v}_a^i-\mathbf{v}_a^j)^2]\,.$$

On the right-hand-side of Eq. (2), the tensor $C_{ab}^{ik\ell}$ becomes

$$(C_{ab}^{ik\ell})_{\mathrm{GP}}=\int d\mathbf{v}F_a^i(\mathbf{v})\,C_{ab}^{k\ell}(\mathbf{v})\,,\quad (C_{ab}^{ik\ell})_{\mathrm{CC}}=C_{ab}^{k\ell}(\mathbf{v}_i)\,,$$

for the Galerkin and center-collocation, respectively. Obtaining the center-collocation tensor $(C_{ab}^{ik\ell})_{\mathrm{CC}}$ is merely a matter of evaluating the RBF tensor $C_{ab}^{k\ell}(\mathbf{v})$ at the collocation points. Evaluation of the tensor $(C_{ab}^{ik\ell})_{\mathrm{GP}}$ for the Galerkin projection is somewhat more complicated, although the result may be potentially be more accurate or robust.

Nevertheless, to maintain simplicity, we focus hereafter on the center collocation-method and omit the CC subscript for brevity. In this case the RBF equations for the full non-linear system become

$$\sum_j M_a^{ij}\mathcal{L}_a^{ij}w_a^j=\sum_{b,k,\ell} w_a^k\,w_b^\ell\,C_{ab}^{ik\ell}\,,\quad\forall\quad i\in 1,2,3,\ldots, \tag{3}$$

where the operator $\mathcal{L}$

$$\mathcal{L}_a^{ij}\doteq\frac{\partial}{\partial t}+\mathbf{v}_a^i\cdot\nabla+\frac{e_a}{\sigma_a^j}\Big[(\mathbf{v}_a^j-\mathbf{v}_a^i)\cdot\mathbf{E}+(\mathbf{v}_a^j\times\mathbf{v}_a^i)\cdot\mathbf{B}\Big]$$

retains the familiar appearance of the Vlasov operator even though the velocity-space has been completely removed from the problem. Note that $\mathcal{L}$ depends explicitly on species a and implicitly on b via the Maxwell equations. In $\mathcal{L}$ we have defined the temperature-like parameter $\sigma_a^i=m_a/(2\gamma_a^i)$ and also dropped some additional terms that arise if the parameters γ_a^i and $\mathbf{v}_a^i$ depend on position. The RBF-kinetic equation (3) describes collisional fluid-like evolution of the weights in time and space. One physically appealing feature of the RBF expansion is that familiar expressions for macroscopic fluid moments retain their intuitive form:

$$\text{Density}\quad n_a=\sum_i w_a^i$$

$$\text{Velocity}\quad n_a\mathbf{V}_a=\sum_i w_a^i\,\mathbf{v}_a^i$$

$$\text{Temperature}\quad \frac{3}{2}n_aT_a=\sum_i w_a^i\left[\frac{3}{2}\sigma_a^i+\frac{1}{2}m_a(\mathbf{v}_a^i-\mathbf{V}_a)^2\right]$$

$$\text{Momentum flux tensor}\quad \mathbf{\Pi}_a=\sum_i w_a^i\,m_a\left[\sigma_a^i\,\mathbb{I}+\mathbf{v}_a^i\mathbf{v}_a^i\right]$$

$$\text{Energy flux}\quad \mathbf{Q}_a=\sum_i w_a^i\,\mathbf{v}_a^i\left[\frac{5}{2}\sigma_a^i+\frac{1}{2}m_a(v_a^i)^2\right].$$

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