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Decoding suprathreshold stochastic resonance with optimal weights

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ABSTRACT

We investigate an array of stochastic quantizers for converting an analog input signal into a discrete output in the context of suprathreshold stochastic resonance. A new optimal weighted decoding is considered for different threshold level distributions. We show that for particular noise levels and choices of the threshold levels optimally weighting the quantizer responses provides a reduced mean square error in comparison with the original unweighted array. However, there are also many parameter regions where the original array provides near optimal performance, and when this occurs, it offers a much simpler approach than optimally weighting each quantizer's response.

1. Introduction

The term stochastic resonance (SR) [1–5] is used to describe phenomena where improvement of transmission or processing of a signal in a nonlinear system is achieved by tuning the noise intensity. Since its origins thirty years ago in the field of geophysical dynamics [1], SR has received considerable attention in a growing variety of systems with various types of signals and performance measures [6–21]. Most SR studies carried out today occur in threshold-based or potential barrier systems where a signal is by itself too weak to overcome a threshold or a potential barrier [6–21], but the presence of noise allows the signal to cross the threshold eliciting a more effective system response. Therefore, subthreshold input signals in threshold-based systems were originally assumed to be a necessary condition for the occurrence of SR.

Interestingly, a form of SR was reported by Stocks [22–24], under the name of suprathreshold SR (SSR), since it operates with signals of arbitrary magnitude, not restricted to weak or subthreshold signals. Notably, SSR is an important extension of SR with potential applications in a range of areas including neural systems. For example, SSR has been considered in ensembles of sensory neurons [25], signal quantizers [41], cochlear implant devices [26] and nonlinear detectors [28]. Moreover, artificial sensors, digital beamforming, biological neurons, cochlear implants and multiaccess communication systems can all be unified under the concept of stochastic pooling networks that manifest the noise-enhanced processing property [32–34]. Due to the variety of scenarios where SSR is observed, a number of performance measures have been considered, for instance, mutual information [22,23,27,29–31], mean square error (MSE) distortion [35,41,43], input–output cross-correlation [35,38], Fisher information [36,39,43] and signal-to-noise ratio [37].

The model studied in [22] that exhibits SSR is effectively a stochastic quantizer, since it converts an analog input signal into a digital output signal with threshold values randomized by noise [40–43]. McDonnell et al. have analyzed SSR in terms of lossy source coding and quantization theory, and examined the optimality of the quantization by using MSE distortion [40–43]. It was shown that the case of all identical threshold values is optimal for sufficiently large input noise, and a bifurcation pattern appears in the optimal threshold distribution with decreasing noise intensity, whether maximizing the mutual information or minimizing the MSE distortion [40–43].

In this paper, we investigate the decoding scheme of a quantized signal in the generic SSR model [22]. We propose a new

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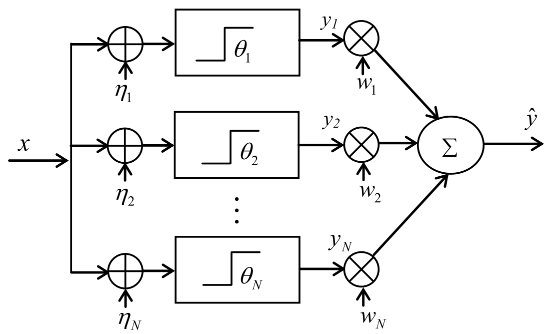

Fig. 1. Weighted summing array of N noisy comparators. It consists of N identical comparators (i.e. single bit quantizers), each operating on a common signal x subject to independent additive noise η_i. The output of each individual comparator, y_i, is multiplied by the weighted coefficient w_i, resulting in the weighted output $w_i y_i$. The overall output, $\hat{y}$, is the sum of the N weighted outputs, i.e. $\hat{y} = \sum_{i=1}^{N} w_i y_i$.

decoding scheme, which we refer to as *optimal weighted decoding*. For different threshold value settings, the MSE distortion curve exhibits the SSR effect as a function of noise level and increased numbers of comparators. We compare the optimal weighted decoding scheme obtained by weighting before summation to that of weighting after summation, by analyzing the MSE distortions of each. The results show that optimal weighting of the binary quantizers' outputs before summation is superior to the case assumed in the original array, where the unweighted binary responses are simply summed. We demonstrate that optimally weighting the responses reduces the MSE distortion between the original input signal and the decoded output signal. However, we also find that there are parameter regions where optimal weighting provides a negligible reduction in mean square error, and in these regions it is therefore beneficial to avoid the additional complexity required in finding the optimal weightings and applying them.

This paper is organized as follows: Section 2 gives mathematical descriptions of optimal weighted decoding for an array of comparators. Section 3 develops the MSE distortion performance of weighted decoding for three examples of threshold setting configuration. Section 4 compares the MSE distortions between the cases of weighting before and after summation. Finally, we present the conclusions and discuss further research directions.

2. Optimal weighted decoding scheme

We here consider the weighted summing array of N noisy comparators, as shown in Fig. 1. All comparators receive the same continuously valued input signal x with standard deviation σ_x. The ith comparator is subject to independent and identically distributed (i.i.d.) additive noise components η_i with standard deviation σ_η, which are independent of the signal x. The output from each comparator, y_i, is unity if the input signal plus the noise is greater than its threshold θ_i, and zero otherwise. The noisy binary output of each individual comparator y_i is then multiplied by the weighted coefficient w_i ($w_i \in \Re$), resulting in the weighted output $w_i y_i$. All weighted outputs are summed to give the overall output $\hat{y} = \sum_{i=1}^{N} w_i y_i$.

When all weighted coefficients w_i $(i = 1, \ldots, N)$ are equal to unity, the model is identical to that studied in [22]. It is effectively a stochastic quantizer [40–43]. The summation of the outputs of all the comparators is a discretely valued stochastic encoding of x, which can take integer values between zero and N. For obtaining reconstructed signal, we need a decoding method to decode the output signal. This is performed by weighting after summation. When the weighted coefficient w_i $(i = 1, \ldots, N)$ is arbitrarily chosen, the model achieves a decoding function that is performed by weighting before summation.

2.1. Wiener linear decoding

Before considering how to optimally weight the quantizer responses, we first review what is known as *Wiener linear decoding*, as studied in [43]. In this case, we introduce y to denote the unweighted sum of the quantizer response, i.e.

$$y = \sum_{i=1}^{N} y_i. \tag{1}$$

It is shown in [43] that, under the condition where all threshold levels are identical and equal to the signal mean, and both the signal and noise have even probability density functions, that $\mathrm{E}[y] = N/2$. Under these conditions, it is of interest to consider how to optimize the MSE between the input signal, x, and a linear decoding of y written in the form

$$\hat{y}_{\mathrm{w}} = \frac{2c}{N} y - c. \tag{2}$$

The result of this operation, $\hat{y}_{\mathrm{w}}$ can be thought of as the reconstructed value of the input signal, with the error between the input x, and the reconstructed output $\hat{y}_{\mathrm{w}}$ being

$$\epsilon = x - \hat{y}_{\mathrm{w}}. \tag{3}$$

It is straightforward to derive the optimal solution for c as

$$c = \frac{N\mathrm{E}[xy]}{2\,\mathrm{var}[y]}, \tag{4}$$

where $\mathrm{var}[y] = \mathrm{E}[y^2] - N^2/4$ is the variance of y [43]. This is known as the Wiener optimal linear decoding scheme for minimizing MSE distortion [44]. The MSE distortion for Wiener decoding can be written as [43]

$$\mathrm{MSE_w} = \mathrm{E}[x^2]\left(1 - \frac{\mathrm{E}[xy]^2}{\mathrm{E}[x^2]\,\mathrm{var}[y]}\right) = \mathrm{E}[x^2](1 - \rho_{xy}^2), \tag{5}$$

where ρ_{xy} is the correlation coefficient between the input signal x and the output y. Equation (5) also shows that the MSE distortion of Wiener decoding scheme is entirely dependant on the correlation coefficient ρ_{xy}.

2.2. Optimal weighted decoding

We now consider the case shown in Fig. 1, where arbitrary multiplicative weightings w_i $(i = 1, \ldots, N)$ are applied to the binary quantizer outputs. We seek to choose the optimal weights, $\mathbf{w}^{\mathrm{o}} = [w_1^{\mathrm{o}}, w_2^{\mathrm{o}}, \cdots, w_N^{\mathrm{o}}]^\top$ under which the MSE distortion between the decoded signal and the input is the minimum. We denote this decoding scheme as *optimal weighted decoding*, and find the optimal weights by applying least squares regression to a data obtained by simulating a sequence of samples from the input signal, and the resulting binary quantizer responses from each sample.

To begin, we introduce a vector $\mathbf{x}$ of size $(K \times 1)$ to denote a sequence of K independent samples drawn from the input signal's probability distribution. We also introduce a matrix $\mathbf{Y}$ of size $(K \times N)$ to denote the N threshold responses for each of the K input samples. We denote an arbitrary vector of weights as $\mathbf{w} = [w_1, w_2, \cdots, w_N]^\top$ and the optimal weights as $\mathbf{w}^{\mathrm{o}}$. Ideally, we desire $\mathbf{w}^{\mathrm{o}}$ to satisfy

$$\mathbf{Y}\mathbf{w}^{\mathrm{o}} = \mathbf{x}. \tag{6}$$

However, for $K > N$ (in practice we desire $K \gg N$), this is an overcomplete system of linear equations, and we therefore follow the standard approach of seeking to find $\mathbf{w}^{\mathrm{o}}$ that minimizes the MSE