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When chaos meets hyperchaos: 4D Rössler model

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ABSTRACT

Chaotic behavior is a common feature of nonlinear dynamics, as well as hyperchaos in high-dimensional systems. In numerical simulations of these systems it is quite difficult to distinguish one from another behavior in some situations, as the results are frequently quite "noisy". We show that in such systems a global hyperchaotic invariant set is present giving rise to long hyperchaotic transient behaviors. This fact provides a mechanism for these noisy results. The coexistence of chaos and hyperchaos is proved via Computer-Assisted Proofs techniques.

1. Introduction

It's only a few decades since famous discovery of Lorenz [1] that deterministic systems can exhibit sensitive dependencies on initial conditions. However, a large number of researchers have been working deeply in the development of theoretical basements needed for the analysis of chaotic systems. A recent cornerstone theoretical result was the Tucker's computer-assisted proof of the existence (and of the mathematical structure) of the Lorenz chaotic attractor [2–4]. Furthermore, it has been shown that these systems reproduce nicely complex behaviors found in real systems of diverse nature [5–11]. Most of the results have been stated in three-dimensional models, where, due to the restricted phase-space only low-dimensional chaos can be observed. The remaining main question is: What changes when higher-dimensional systems are analyzed?

Chaotic systems are characterized by (at least) one direction of exponential spreading. A common way to detect this circumstance is by calculating the maximum Lyapunov exponent [12] of the orbit. If it is positive, the orbit exhibits sensitive dependence on initial conditions, and this is a standard indication of chaotic behavior (we remark that a positive Lyapunov exponent is not always an indication of chaos, as shown, for instance, in [13–16]). If the number of directions of spreading is greater than one, the behavior of the system is hyperchaotic [17]. To detect this situation it becomes necessary to calculate more Lyapunov exponents and to determine how many of them are positive. Note that for continuous autonomous dynamical systems, chaos can appear in systems with dimension greater than or equal to three.

The behavior (and therefore its analysis) of hyperchaotic systems is much more complicated than the case of systems with just a single positive Lyapunov exponent. There are two main reasons, firstly the need for a fourth dimension to the appearance of hyperchaos, which makes some tools of analysis for three-dimensional chaotic models not valid; on the other hand, the existence of more than one direction of spreading allows the system undergo a broader spectrum of bifurcations. However, in practical applications it is necessary to model problems with dimension higher than three, in some of which hyperchaotic behavior appears like in mathematical models of electroencephalograms, chemical systems, electronic circuits [18–21] and in most of the networks of basic continuous systems, as coupled Lorenz or Rössler systems [22]. Besides, note that contemporary numerical weather prediction schemes are based on ensemble forecasting. Ensemble members are obtained by taking different (perturbed) models started with different initial conditions, an example of such kind of systems is the Lorenz-96 model [23]. In all these kinds of systems the appearance of hyperchaos is quite natural due to their dimension. In addition, this high dependence on initial conditions experienced by hyperchaotic systems has practical applications, such as encryption of information [24,25].

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That is why in the last two decades many articles have appeared in which the authors study dynamical systems with hyperchaotic behavior [26–30]. Many of them focus on the transition from chaotic to hyperchaotic behavior. The problem is that the numerical study of these systems is sometimes no clear at all, giving really a confuse analysis about whether the system is chaotic or hyperchaotic [31]. Therefore, one of the main goals of this paper is to study in detail the main reason of why most of these studies really fail in giving a clear picture of what happens in the system. We show that the main behavior of these systems is hyperchaotic, but it can be a transient behavior or an attracting one. This duality is rigorously established via Computer-Assisted Proofs (CAP) techniques and it gives a mechanism for the "noisy" simulations in many studies [31].

The paper is organized as follows. In Section 2 we present several numerical simulations on the 4D Rössler model to study the appearance of chaotic and hyperchaotic behaviors and how the results depend on the way of computing the Lyapunov exponents. In Section 3 we give the basic steps of a computer-assisted proof of the coexistence of chaotic and hyperchaotic behavior, giving in some situations the existence of long hyperchaotic transients that may give rise to "noisy" numerical simulations. Finally, in Section 4 we present some conclusions.

2. Chaos and hyperchaos: numerical studies

Our first question is to study the detection of the different behaviors (regular, chaotic or hyperchaotic) of the dynamical models. The usual approach is to calculate two or more dominant Lyapunov exponents and determine how many of them are positive. Along this paper, we will use, as paradigmatic example, the well-known 4D Rössler model [17], given by:

$$\begin{cases} \dot{x} = -(y+z), \\ \dot{y} = x + ay + w, \\ \dot{z} = b + xz, \\ \dot{w} = -cz + dw, \end{cases} \tag{1}$$

where we fix the values of parameters $b = 3.0$ and $d = 0.05$, and we allow to change the values of a and c. This model was the first model where it was shown the existence of hyperchaotic behavior.

In Fig. 1 we present two biparametric plots showing the different behaviors based on the Lyapunov exponents computed using the algorithm of Wolf et al. [32]. The only difference in the simulations is that the lower picture is done considering a transient time 3×10^4 before computing the exponents. In the simulations we have differentiated the cases of having two large positive Lyapunov exponents (strong hyperchaos) with the case of having two positive values but one of them quite small (weak hyperchaos). The colors in the figure determine the different behaviors detected in the simulations. White represents a limit cycle, maximum Lyapunov exponent $\lambda_1 = 0$ and the others $\lambda_{2,3,4} < 0$; blue for torus, $\lambda_{1,2} = 0$ and $\lambda_{3,4} < 0$; red for chaotic, $\lambda_1 > 0$, $\lambda_2 = 0$ and $\lambda_{3,4} < 0$; green for weak hyperchaos (see Fig. 3), $0.05 > \lambda_1 > \lambda_2 > 0$, $\lambda_3 = 0$, $\lambda_4 < 0$; brown for strong hyperchaos, $\lambda_1 > 0.05 > \lambda_2 > 0$, $\lambda_3 = 0$, $\lambda_4 < 0$. Comparing both pictures, we can see how the upper picture is completely dominated by brown color, representing hyperchaotic behavior. Note that in almost all the results we have a "noisy" picture without giving a clear study of the real behavior, especially in the upper plot. This situation appears also in most of the simulations in literature [31]. In contrast, in the lower picture, wherein we have used the transient time, those structures that were hardly visualized in the upper picture, now appear in a clearer way, but still some "noisy" patterns appear. The integration time (without considering the transient time) used in both

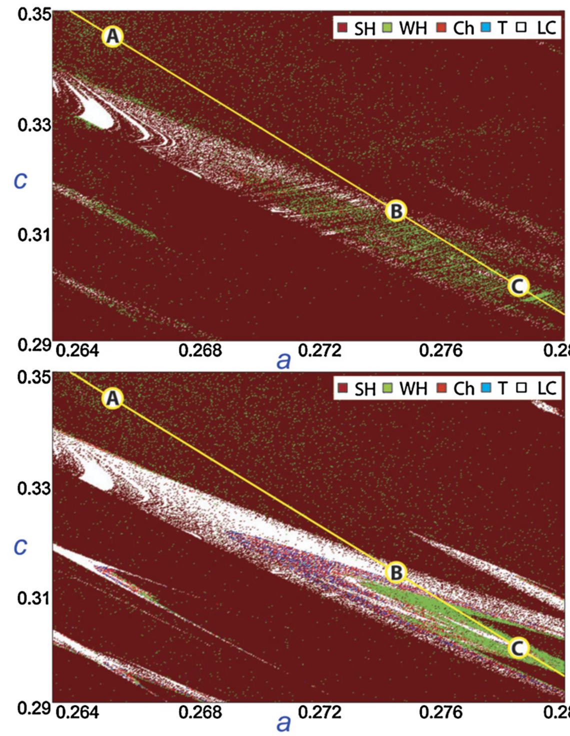

Fig. 1. Lyapunov exponents biparametric plots showing periodic (limit cycles, LC), quasiperiodic (torus, T), chaotic (Ch), weak-hyperchaotic (WH) and strong-hyperchaotic (SH) behaviors. (Top) without transient time in the simulations and (bottom) with transient time. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

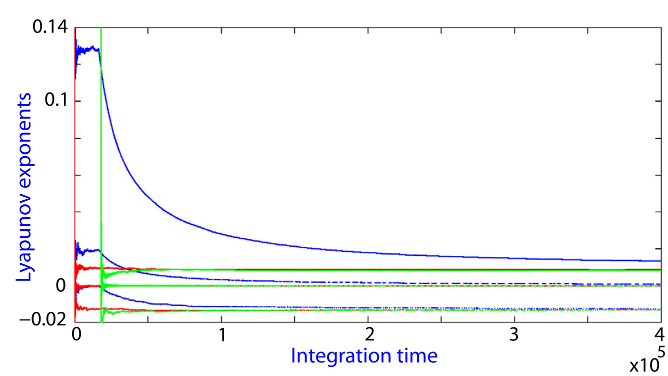

Fig. 2. Time evolution of the Lyapunov exponents depending on using or not transient time and depending on the initial conditions. Red, without transient time but with initial conditions that go directly to the chaotic attractor; blue, without transient time, with initial conditions that go first close to the hyperchaotic saddle; green, with transient time. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

pictures is 3000, the first question is whether the upper picture, at least, correctly identifies the behavior of the system at that final moment. Fig. 2 shows how the calculation of exponents keeps memory of past behavior, taking time to recognize behavior changes [33]. This tells us that if we want to classify the behavior of the system at any given time, we must try to start the calculation of Lyapunov exponents from a time at which the system experiences such behavior. If we want to study the type of attractor which goes into the dynamics of the system, then we have to consider a sufficiently long transient time. We can see that the second picture of Fig. 1 still shows "noise", so that higher transient and/or integration time is necessary to get an absolutely clear picture. In a more detailed study [34], it has been found that a