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On vertex conditions for elastic systems

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ABSTRACT

In this paper vertex conditions for the differential operator of fourth derivative on the simplest metric graph – the Y-graph, – are discussed. In order to make the operator symmetric one needs to impose extra conditions on the limit values of functions and their derivatives at the central vertex. It is shown that such conditions corresponding to the free movement of beams depend on the angles between the beams in the equilibrium position.

1. Motivation

Quantum graphs – differential operators on metric graphs – form a well-established formalism to model wave propagation in thin channels and other physical systems where interesting phenomena occur in a neighborhood of a system of low-dimensional manifolds [1,12,21]. It appears that ordinary differential equations on graph edges coupled by certain vertex conditions give a rather good approximation of wave propagation in thin domains. Intensive research is devoted to understanding which particular vertex conditions give the best approximation. The set of all possible vertex conditions can be described using different mathematical languages [15,10,11,14,16,2]. One may expect that the geometry (first of all the angles between the edges) of the vertex should be reflected in these conditions. To study this question it is enough to consider the star graph and in this article we are going to restrict our consideration to the star graph Γ formed by 3 semi-infinite edges $[x_j, \infty)$, $j = 1, 2, 3$. There is just one vertex $V = \{x_1, x_2, x_3\}$.

In physics it is common to use quadratic forms instead of operators, since often it is easier to check that a quadratic (sesquilinear) form is semi-bounded and closed instead of proving that an operator is self-adjoint. The quadratic form corresponding to the Laplacian on Γ is given by the integral of $|u'(x)|^2$ plus possibly an additional term coming from the vertex. This additional term describes the interaction between the waves at the vertex and should be absent if one is looking for conditions corresponding to the free motion. We end up with the quadratic form

$$Q[u, v] = \int_\Gamma u'(x)\overline{v'(x)}dx = \sum_{j=1}^{3} \int_{x_j}^{\infty} u'(x)\overline{v'(x)}dx. \tag{1}$$

This form is positive and closed on the set of functions from the Sobolev space $W_2^1(\Gamma \setminus V) = \bigoplus_{j=1}^3 W_2^1(x_j, \infty)$. If no further conditions at the vertex are assumed, then the corresponding self-adjoint operator is the Laplace operator $L_{II} = -\frac{d^2}{dx^2}$ defined on the functions from $W_2^2(\Gamma \setminus V)$ satisfying Neumann conditions $u'(x_j) = 0$. The three edges are then independent of each other. Therefore the only way to introduce coupling between the edges is through restricting the domain of the quadratic form by certain conditions on the limiting values of the function at the vertex (to ensure that the operator is local). Since the functions from W_2^1 are continuous, but their derivatives are not necessarily continuous, these vertex conditions may involve only the values of the function at the end points $u(x_j)$. Here and in what follows we use the limiting values of functions and their derivatives from inside the edges defined as

$$u^{(n)}(x_j) := \lim_{\epsilon \searrow 0} u^{(n)}(x_j + \epsilon). \tag{2}$$

If all endpoints are equivalent, then it is natural to impose the continuity condition

$$u(x_1) = u(x_2) = u(x_3). \tag{3}$$

The self-adjoint operator corresponding to the quadratic form (1) defined on functions from $W_2^1(\Gamma \setminus V)$ satisfying (3) is precisely the Laplace operator L_{II} with the domain given by the standard conditions

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$$\begin{cases} u \text{ is continuous at the vertex } V; \\ \sum\limits_{x_j \in V} u'(x_j) = 0. \end{cases} \tag{4}$$

Another possibility to restrict the quadratic form is to introduce Dirichlet conditions

$$u(x_j) = 0, \; x_j \in V \tag{5}$$

instead of (3). The corresponding operator is self-adjoint but there is no coupling between the edges.

We have seen that the derived vertex conditions do not contain any information about the geometry of the junction, especially the angles between the edges are not reflected. In what follows we turn to problems of elasticity and will show how the geometry of the junction may be reflected in vertex conditions.

2. Elasticity

Elasticity problems are usually described by fourth order differential operators [19]. Consider therefore the differential operator $L_{IV} := \frac{d^4}{dx^4}$ first on the interval $[x_0, \infty)$ and later on the graph Γ. The corresponding wave equation

$$\frac{\partial^4}{\partial x^4} u = -\frac{\partial^2}{\partial t^2} u$$

can be used to describe the dynamics of small deflection of beams. Thus the operator L_{IV} in $L^2(x_0, \infty)$ corresponds to a single long beam. The operator can be made self-adjoint by choosing appropriate boundary conditions at the origin. For example, in the case when the end point of the beam is clamped the corresponding self-adjoint boundary condition is

$$u(x_0) = u'(x_0) = 0.$$

And in the case when the end point is free we can impose the following condition to make the operator self-adjoint

$$u''(x_0) = u'''(x_0) = 0. \tag{6}$$

One can describe all possible boundary conditions at the end point x_0 by the following families of matching conditions (see [23]):

0) $u(x_0) = u'(x_0) = 0,$

i) $u(x_0) = 0, \qquad u''(x_0) = \gamma_1 u'(x_0),$

ii) $u'(x_0) = \gamma_3 u(x_0), \qquad -u'''(x_0) + \overline{\gamma_3} u''(x_0) = \gamma_2 u(x_0),$

iii) $\begin{cases} u''(x_0) = \gamma_3 u(x_0) + \gamma_1 u'(x_0) \\ u'''(x_0) = -\gamma_2 u(x_0) - \overline{\gamma_3} u'(x_0), \end{cases}$

where γ_1, γ_2 are arbitrary real numbers and γ_3 is an arbitrary complex number. This formalism can be carried out to the case where several beams are connected together at one junction. To describe all possible vertex conditions leading to self-adjoint operators one may use either von Neumann theory [22], boundary triples [8] or scattering matrix approach [12–14]. Our goal here is not to carry out such description, but to derive conditions corresponding to the free dynamics of beams.

Consider first the case where just two beams are connected. Assume that the equilibrium position corresponds to the case when the two beams form a line. Such beams may be parameterized as $(-\infty, x_0]$, and $[x_0, \infty)$. The quadratic form corresponding to the free motion is given by

$$\begin{aligned} Q[u, v] &= \int_{-\infty}^{\infty} u''(x)\overline{v''(x)}dx \\ &= \int_{-\infty}^{x_0} u''(x)\overline{v''(x)}dx + \int_{x_0}^{+\infty} u''(x)\overline{v''(x)}dx. \end{aligned} \tag{7}$$

As before it is natural to assume that u is continuous at x_0, otherwise the beams do not touch each other:

$$u(x_0 - 0) = u(x_0 + 0). \tag{8}$$

The beams are free, no external force is applied if the graph of $u(x)$ does not form any angle at x_0, *i.e.* the first derivative is continuous:

$$u'(x_0 - 0) = u'(x_0 + 0). \tag{9}$$

The quadratic form on the domain of functions from $W_2^2(\mathbb{R} \setminus x_0)$ satisfying conditions (8) and (9) is semi-bounded and closed. Let us calculate the self-adjoint operator corresponding to this quadratic form. First of all one needs to determine the set of all u, for which the quadratic form $Q[u, v]$ is a bounded linear functional with respect to v. Taking v from $C_0^\infty(\mathbb{R} \setminus x_0)$ we see that the second derivative of the function u'' should be from $L_{2,\text{loc}}(\mathbb{R} \setminus x_0)$. In other words $u \in W_2^4(\mathbb{R} \setminus x_0)$ and one may carry out integration by parts:

$$\begin{aligned} Q[u, v] = & \int_{-\infty}^{x_0} u^{(iv)}(x)\overline{v(x)}dx + \int_{x_0}^{+\infty} u^{(iv)}(x)\overline{v(x)}dx \\ & - u''(x_0+0)\overline{v}'(x_0+0) + u'''(x_0+0)\overline{v}(x_0+0) \\ & + u''(x_0-0)\overline{v}'(x_0-0) - u'''(x_0-0)\overline{v}(x_0-0) \\ = & \int_{-\infty}^{x_0} u^{(iv)}(x)\overline{v(x)}dx + \int_{x_0}^{+\infty} u^{(iv)}(x)\overline{v(x)}dx \\ & - (u''(x_0+0) - u''(x_0-0))\overline{v'}(x_0) \\ & + (u'''(x_0+0) - u'''(x_0-0))\overline{v}(x_0), \end{aligned} \tag{10}$$

where on the last step we used that the function v satisfies (8) and (9). The integral terms are bounded linear functionals with respect to v, while the functionals

$$v \mapsto v(x_0) \text{ and } v \mapsto v'(x_0)$$

are not. It follows that the coefficients in front of these functionals must be equal to zero

$$\begin{cases} u''(x_0 - 0) &= u''(x_0 + 0), \\ u'''(x_0 - 0) &= u'''(x_0 + 0). \end{cases} \tag{11}$$

The operator $\frac{d^4}{dx^4}$ defined on the set of functions from $W_2^4(\mathbb{R} \setminus x_0)$ satisfying matching conditions (8), (9) and (11) is the self-adjoint operator corresponding to the quadratic form. As in the case of Laplacian the vertex x_0 can be removed and the two edges may be substituted by one edge $(-\infty, \infty)$.

Our goal is to understand how the matching conditions (8), (9), (11) can be generalized to the case when several beams are joined at one vertex, having different angles in-between in the equilibrium position. The function u describing small deflections of the system from the equilibrium can be considered as a function on the graph Γ made up of three half-lines connected at the vertex V and making angles α, β and γ, none of which is equal to 0 or π (see Fig. 1). The special cases when one of the angles is equal to 0 or π is considered in Section 4. The Hilbert space is $L_2(\Gamma) := \bigoplus_{i=1}^3 L_2[x_j, \infty)$ and we are interested in vertex