Physics Letters A 379 (2015) 1934–1937

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

# Impact of third-order dispersion on nonlinear bifurcations in optical resonators

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ARTICLE INFO

*Article history:*
Received 27 March 2015
Accepted 16 April 2015
Available online 21 April 2015
Communicated by V.M. Agranovich

ABSTRACT

It is analytically shown that symmetry breaking, in dissipative systems, affects the nature of the bifurcation at onset of instability resulting in transitions from super to subcritical bifurcations. In the case of a nonlinear fiber cavity, we have derived an amplitude equation to describe the nonlinear dynamics above threshold. An analytical expression of the critical transition curve is obtained and the predictions are in excellent agreement with the numerical solutions of the full dynamical model.



## 1. Introduction

Dissipative structures arise in many different fields of nonlinear science [1]. They are stable patterns that arise far from equilibrium in dissipative systems. They exist because of the interplay between diffusion/diffraction and nonlinearity on one hand and between losses and an external source on the other hand. It was shown that such structures are significantly modified when a symmetry breaking is present in the system, i.e., odd-order spatial (for diffractive/diffusive systems) or temporal (for dispersive/temporal systems) derivatives leading to convective drift, or walk off terms depending on the specific physical situations. The effect of convective terms on dissipative structures has attracted a lot of interest in hydrodynamics [2], plasma physics [3], traffic flow [4] and nonlinear optics [5]. The main focus has been on the induced drift and the resulting convectively or absolutely unstable regimes [6,7]. More recently, there have been some studies on the nonlinear symmetry breaking induced by those terms in nonlinear optical dissipative systems [8,9]. They showed that albeit the linear nature of the broken symmetry, the nonlinearity of the system is affected, leading to spectral asymmetry and power dependent velocities of the traveling waves.

Here, we report on an important aspect of symmetry breaking that has not yet been investigated that drastically affects nonlinear bifurcations occurring in dissipative systems. We show that the time reversal symmetry breaking induced by third-order dispersion term drastically impacts the nature of the bifurcation from supercritical to subcritical and vice versa. From physical point of view, this leads to the appearance of a bistable regime between modulated and unmodulated solutions that can lead to the formation of localized structures. Here we perform, to the best of our knowledge, the first analytical study of the effect of convective terms on the nonlinear bifurcation in the case of modulated solutions appearing in optical resonators. We consider a passive nonlinear resonator whose dynamics is well described, in the mean field approximation, by the well known Lugiato–Lefever (LL) equation [10]. In a recent study, we analytically and experimentally investigated the spectral asymmetry induced by a convective term [9]. We showed that the third-order dispersion (TOD) induced an asymmetry in the intensities of high-order harmonics resulting in a transition from symmetric to asymmetric dissipative structures. In this work, we show how the symmetry breaking actually modifies the intrinsic nonlinearity of the system. More precisely, we show that it affects the bifurcation nature, at onset of the instability, leading to *a transition from a sub to a supercritical bifurcation* which drastically impacts the subsequent dynamics above threshold. The analytical description of this bifurcation transition, based on the amplitude equation of the passive optical cavity, demonstrates an original dependence of the nonlinear saturation term upon the symmetry breaking term (here the third-order dispersion term).

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http://dx.doi.org/10.1016/j.physleta.2015.04.027
0375-9601/© 2015 Published by Elsevier B.V.

## 2. The model

We start from the dimensionless mean-field equation describing nonlinear resonators near the zero-dispersion wavelength [11]:

$$\partial_t E(t,\tau) = \left[-1 + \mathrm{i}(|E(t,\tau)|^2 - \Delta) - \mathrm{i}\eta\partial_\tau^2 + d_3\partial_\tau^3\right] E(t,\tau) + S, \tag{1}$$

where $S$ and $E$ are the normalized slowly varying envelopes for pump and signal fields respectively, $\Delta$ is the normalized cavity detuning, $\eta$ is the sign of the second-order dispersion term (SOD) and $d_3$ is the normalized third-order dispersion coefficient. $t$ and $\tau$ correspond respectively to the slow and fast time [12]. Details on the normalization can be found in [13,11]. Both convective and absolute instabilities have been recently reported for the steady-state solution $E_s$ satisfying the equation $S = [1 + i(\Delta - I_s)]E_s$ where $I_s = |E_s|^2 = E_s^2$ ($E_s$ is taken to be real) [14]. In this paper, we devote our attention to the monostable case (i.e. $\Delta < \sqrt{3}$) with anomalous dispersion where Eq. (1) exhibits a bifurcation at $I_s = 1$, corresponding to a pump power $X_s = |S|^2 = 2 - 2\Delta + \Delta^2$. The system then evolves towards modulated solutions characterized by a frequency $\Omega_c = \sqrt{(\Delta - 2)/\eta}$ and a wave vector $\kappa_c = -d_3\Omega_c^3$ [14]. When $d_3 = 0$, Eq. (1) reduces to the well-know LL equation describing nonlinear spatial cavities [10]. It has been shown that the modulated stationary solutions of that equation can be analytically calculated by the standard method of bifurcation theory [15,16]. From this study comes the well-known transitional detuning $\Delta_t = 41/30$, characterizing the passing from a supercritical bifurcation, where the modulated solutions appear above threshold for pump powers higher than $X_s$ and are stable, to a subcritical bifurcation where the modulated solutions appear above threshold for pump powers lower than $X_s$ and are unstable.

## 3. Analytical approach

In order to study the effect of the third-order dispersion on the stationary modulated solutions and the transitional detuning, we perform the same multi-scale analysis right above the instability threshold but now including third-order dispersion. We start by expanding the variables in multiple orders of a small parameter $\varepsilon$, defined as $\varepsilon^2 = I_s - 1$, that is the distance from the instability threshold. The envelope of the electric field is rewritten in terms of the amplitudes $a_k$, defined by $E = E_s + \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 a_3 + \ldots$. Following the approach of [15,16], and taking into account the gain spectrum of the instability [12] we expand the slow time $t$ and the fast time $\tau$. We introduce the new times: $T_0 = t$, $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t$, $\tau_0 = \tau$ and $\tau_1 = \varepsilon\tau$ so that the corresponding temporal derivatives become $\partial_t = \partial_{T_0} + \varepsilon\partial_{T_1} + \varepsilon^2\partial_{T_2}$ and $\partial_\tau = \partial_{\tau_0} + \varepsilon\partial_{\tau_1}$. We then assume that the amplitudes $a_k$ is the sum of quasi-monochromatic waves written in the form $a_1 = \left(A_1 \mathrm{e}^{\mathrm{i}(\Omega_c\tau_0 + \kappa_c T_0)} + A_1^* \mathrm{e}^{-\mathrm{i}(\Omega_c\tau_0 + \kappa_c T_0)}\right)$ where $A_1$ and its complex conjugate $A_1^*$ are slowly varying amplitudes, and $a_k = D_k + A_k^+ \mathrm{e}^{\mathrm{i}(\Omega_c\tau_0 + \kappa_c T_0)} + A_k^- \mathrm{e}^{-\mathrm{i}(\Omega_c\tau_0 + \kappa_c T_0)} + C_k^+ \mathrm{e}^{2\mathrm{i}(\Omega_c\tau_0 + \kappa_c T_0)} + C_k^- \mathrm{e}^{-2\mathrm{i}(\Omega_c\tau_0 + \kappa_c T_0)}$ with $k = 2, 3$. This form of $a_1$ is justified by the fact that right above the instability threshold, the gain exceeds unity only in the vicinity of $\Omega \approx \pm\Omega_c$, while for the second- and third-order corrections, contributions at $0, \pm 2\Omega_c$ appear because of the nonlinear interactions. By substitution of the above expansions in Eq. (1), we obtain a hierarchy of equations for the successive orders of $\varepsilon$. The evolution of $A_1$ is described by the following equation, obtained as a consequence of a solvability condition [15, 16] at the third order:

$$\partial_t A + 3d_3\Omega_c^2\partial_\tau A = (I_s - 1)A + (2\Omega_c^2 + 3\mathrm{i}d_3\Omega_c)\partial_\tau^2 A + (s_1 + \mathrm{i}s_2)|A|^2 A, \tag{2}$$

where we have set $A = \varepsilon A_1$ and the parameters are defined as

$$s_1 = 24\frac{2G+3}{G^2} + 4\frac{G^2(1-2G) + H^2(2G-3)}{(G^2-H^2)^2 + 4H^2}, \tag{3}$$

$$s_2 = \frac{4H[2(1-2G) + G^2 - H^2]}{(G^2-H^2)^2 + 4H^2}, \tag{4}$$

with

$$G = 3(\Delta - 2)$$

$$H = -6d_3\Omega_c^3$$

This equation of complex Ginzburg–Landau type describes the time evolution of the Stokes wave (fundamental mode) above threshold. First, note that, in absence of TOD ($\beta_3 = 0$), three terms in Eq. (2) disappear since $d_3 = 0$ and $s_2 = 0$. As a result, the presence of TOD drastically impacts the dynamics by introducing drift and diffraction effects (terms in Eq. (2) with $d_3$) together with a nonlinear frequency modulation (term with $s_2$). More importantly, the presence of TOD affects the nature of the bifurcation as can be seen from Eq. (3) since the nonlinear coefficient $s_1$ can change sign as it will be shown below. This clearly demonstrates how the symmetry breaking introduces a rich and a complex dynamics. However, in what follows, we only investigate the important dynamics resulting from transitions between super- and sub-critical bifurcations. Before proceeding further, and to give a complete analytical description above threshold, let us resolve the system up to the third order. After lengthy but straightforward calculations, we find the following analytical expression for the dissipative structures:

$$E(t,\tau) = D + A^+ \mathrm{e}^{\mathrm{i}(\Omega_c\tau + (\kappa_c+\kappa)t)} + A^- \mathrm{e}^{-\mathrm{i}(\Omega_c\tau + (\kappa_c+\kappa)t)} + C^+ \mathrm{e}^{2\mathrm{i}(\Omega_c\tau + (\kappa_c+\kappa)t)} + C^- \mathrm{e}^{-2\mathrm{i}(\Omega_c\tau + (\kappa_c+\kappa)t)}, \tag{5}$$

with

$$D = E_s + \frac{12|A_{st}|^2}{G^2}\left[(2G+3) + 3\mathrm{i}(G+1)\right], \tag{6}$$

$$A^+ = (1+\mathrm{i})|A_{st}|\,(1 - MH + \mathrm{i}N), \tag{7}$$

$$A^- = (1+\mathrm{i})|A_{st}|\,(1 + MH + \mathrm{i}N), \tag{8}$$

$$C^+ = 2|A_{st}|^2\frac{1 + \mathrm{i} - (2+\mathrm{i})(G-H)}{G^2 - H^2 - 2\mathrm{i}H}, \tag{9}$$

$$C^- = 2|A_{st}|^2\frac{1 + \mathrm{i} - (2+\mathrm{i})(G+H)}{G^2 - H^2 + 2\mathrm{i}H}, \tag{10}$$

$$\kappa = s_2|A_{st}|^2, \tag{11}$$

$$|A_{st}|^2 = (1 - I_s)]/s_1, \tag{12}$$

$$M = |A_{st}|^2\frac{12H - 20GH}{(G^2-H^2)^2 + 4H^2}, \tag{13}$$

$$N = Is - 1 + |A_{st}|^2\left(3 + 12\frac{7G+9}{G^2} + \frac{(6-10G)(G^2-H^2)}{(G^2-H^2)^2 + 4H^2}\right), \tag{14}$$

where $|A_{st}|$ and $\kappa$ are the amplitude and the nonlinear correction to the wavevector of the nonlinear dissipative solution of Eq. (2). Here we purposely neglected the dependence of the spectral amplitudes in the fast time $\tau$ as we are only interested in the leading contribution of each harmonics.

## 4. Third-order dispersion and the nature of the bifurcation

Nonlinear dynamics above threshold is mainly ruled by the last nonlinear term of Eq. (2). The complex coefficient $s_1 + is_2$ of this nonlinear term is of a crucial importance in the above threshold