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Constructing analytic approximate solutions to the Lane–Emden equation

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ABSTRACT

We derive analytic approximations to the solutions of the Lane–Emden equation, a basic equation in Astrophysics that describes the Newtonian equilibrium structure of a self-gravitating polytropic fluid sphere. After recalling some basic results, we focus on the construction of rational approximations, discussing the limitations of previous attempts, and providing new accurate approximate solutions.

1. Introduction

Polytropic fluid sphere models are ubiquitous in Astrophysics. They have been instrumental in the development of stellar structure theory [1], as well as in the investigation of the dynamics of spherical galaxies and star clusters [2]. Gaining insight into their equilibrium and stability properties is therefore an important task that has attracted, and still attracts much interest.

Polytropic models are characterized by a simple equation of state, $p \propto \rho^{(n+1)/n}$, with p and ρ the fluid pressure and density, respectively, and n the so-called polytropic index. In an isotropic configuration, the Newtonian, hydrostatic equilibrium structure of the fluid sphere is then determined by a second order, generally nonlinear ordinary differential equation for the gravitational potential,

$$y'' + \frac{2}{x}y' = -y^n \,, \tag{1}$$

known as the Lane–Emden equation [LEE hereafter; here $y^n = \rho(x)/\rho(0)$, x is a scaled radial coordinate, and the prime denotes derivation with respect to x]. The problem is completed by the boundary conditions

$$y(0) = 1, \qquad y'(0) = 0 \,, \tag{2}$$

which ensure regularity at the sphere center. For $0 \le n < 5$, the solutions of the boundary value problem (1)–(2) decrease monotonically with x and vanish at a finite radius x_1 (the star radius, in a stellar context), which is a rapidly increasing function of n ($x_1 \to \infty$ for $n \to 5$).

Exact solutions to the LEE are only known for the linear cases $n = 0, 1$, and for $n = 5$. For other values of n, well-known numerical methods for initial value problems may be used to compute accurate approximate solutions. Reference results have been obtained in [3] and [4], through Runge–Kutta integrations, and in [5], using the Chebyshev pseudospectral method. Analytic approximations have also been sought; classical works are [6], focusing on rational approximations; [7], in which a sophisticated functional ansatz was developed; and [8], where the delta-perturbation method was used to derive an approximation for $x_1(n)$.

In the last decade, the search for approximate solutions to the LEE has produced many papers (see, e.g., the list given in the introduction of [9]), but, apparently, few useful results. A problem is that most of these works restrict to the interval $[0, 1]$, denoted as the "core region" in astrophysical contexts, even though the radial ranges of interest are typically much larger. Consider, for example, the $n = 3$ polytrope, which provides a reasonable description of the Sun's structure, and is consequently widely used as a test case: since its boundary is at $x = x_1 \simeq 6.897$, a useful approximation for the structure of this polytrope should cover a range about *seven* times larger than the core region.

Moreover, in the core region, approximate solutions of any desired accuracy can be easily constructed using conventional Taylor series expansions about the origin, because the convergence range of these series is always significantly larger than unity (see [10]). It is therefore unclear why so many papers in recent years have focused on using more complicated approaches (Adomian decomposition, Homotopy analysis method, Boubaker polynomials, among others; see [9] and references therein) to derive alternative approximations over $[0, 1]$. Often in these papers important works on the properties of series solutions to the LEE, such as [10] and [11], are not cited, and a detailed comparison with relevant previous work is lacking.

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This provided motivation for the present Letter, whose first objective is to recall some basic results that should be taken into account, and used as a reference where appropriate, by anybody seeking new approximate solutions to the LEE. We shall then focus on the construction of rational approximations, clarifying the limitations of previous attempts, and deriving some new, simple and accurate approximate solutions to (1)–(2).

2. Some basic results

a) *Exact solutions.* It is said in [9] that "only the cases $n = 0$, $n = 1$ and $n = 5$ can be solved analytically...". This is probably true, but, as far as we know, it has not yet been proven. It was stated in [12], without a proof, that application of the Lie group analysis shows that (1) is nonintegrable in a closed form for other values of n, because its Lie algebra is zero-dimensional. But then, it was also noted that there are some – albeit rare – cases in which a zero-dimensional Lie algebra does not preclude integrability. Thus, it would seem more prudent to say that the cases $n = 0, 1, 5$ are the only ones that are currently known to be analytically solvable.

b) *Scaling.* In some works, as for example in [13], an apparent generalization of the problem was considered, with the boundary condition on y given by

$$y(0) = a\,, \tag{3}$$

a being a positive constant. It is readily seen, however, that the scaling

$$y = a\tilde{y}, \qquad x = \tilde{x}/a^{(n-1)/2}, \tag{4}$$

maps (1) into an equation of the same form for $\tilde{y}(\tilde{x})$, with $\tilde{y}(0) = 1$, thus reducing the problem to the standard one. Most of the figures of [13] are just illustrations of this scaling; for example, the only difference between the solutions displayed in Figs. 7, 8, and 9 is a scaling factor a for the y-axis and a $1/a$ factor for the x-axis, in agreement with (4).

c) *Series solutions.* Taylor series expansions for y about the origin (up to the x^{10} power) were given in [13], for several values of n. Those series are special cases of the well-known general expansion

$$\begin{aligned} y \simeq 1 &- \frac{1}{3!}x^2 + \frac{n}{5!}x^4 - \frac{n(8n-5)}{3\times 7!}x^6 \\ &+ \frac{n(122n^2-183n+70)}{9\times 9!}x^8 \\ &- \frac{n(5032n^3-12\,642n^2+10\,805n-3150)}{45\times 11!}x^{10} + \dots \end{aligned} \tag{5}$$

(see, e.g., [4] and references therein). Analytic calculation of higher order terms in (5) is cumbersome, but, when needed, such terms can be easily obtained numerically. We have computed some of them using the stable, coupled recurrence relations for y and ρ given in [10], with the purpose of estimating the accuracy of (5), truncated at the x^{10} term, in the core region. We find that, for $n = 1, 2, 3, 4$, the values of $y(1)$ are correct to $9, 5, 4$, and 3 decimal digits, respectively. Accuracy is of course higher at smaller values of x. Thus, up to $n = 4$, the first six terms in the series expansion (5) yield sufficient accuracy in the core region for most practical purposes.

We note that, since the Taylor series expansion converges in the core region, and can be easily computed with high accuracy, it *should* be used as a benchmark for any alternative approximation over $[0, 1]$.

d) *Convergence of the series solutions.* It has long been known that, for n large enough, the Taylor series expansions about the origin do not cover the whole radial extent of the star (see, e.g., [14] and [15]). More recently, the convergence radius x_s of the series expansion was accurately determined (see [10] and [11]) for several values of n, through non-trivial numerical computations. It was found that x_s is a decreasing function of n, and that the expansion converges over the whole radial extent of the star only for n smaller than about 1.9. For larger values of n, x_s becomes a fraction of x_1: x_s/x_1 is less than $2/5$ for $n = 3$, and only about $2/15$ for $n = 4$. This behavior results from the presence of singularities in the complex plane that were investigated in detail in [11]. Both in [10] and in [11] it was also shown that the singularities may be transformed away through appropriate changes of independent variable. The expansions in the new variables do converge up to the star boundary, albeit quite slowly (very slowly for $n > 3$).

e) *Other approximations.* Despite the long history of the subject, few useful, alternative analytic approximations have been derived that cover the whole radial range. The one constructed in [7] is accurate, and in principle holds for any n, but has a complicated structure, with three coefficients to be fitted, case by case, and an arbitrarily chosen function; optimal coefficients were only computed for $n = 0.5, 1, 1.5, 2, 3$, and for some other n values larger than 5 (see Table 1 of [7]). The $(2, 2)$ Padé approximant computed in [6] was shown to be accurate for $0 \le n \le 2.5$, but its behavior for larger n is unclear. Approximations of a different form were derived in [10], which require a priori knowledge of both x_1 and $y'(x_1)$. The coefficients of these approximations were tabulated for integer and half-integer values of n, in the range $1 \le n \le 4$.

3. Rational approximations

A well-known technique for extending the accuracy of the series expansions beyond their radius of convergence is that of the Padé approximants, which are rational approximations constructed from the Taylor series (see, e.g., [16]). In the context of the LEE, this approach was first pursued in [6], where the first two diagonal Padé approximants, $y_n^{(1,1)}$ and $y_n^{(2,2)}$, were computed. The second Padé approximant, which results from imposing the first four terms in the Taylor series expansion around the origin, was written as

$$y_n^{(2,2)} = \frac{a_1 + a_2x^2 + a_3x^4}{b_1 + b_2x^2 + b_3x^4}\,, \tag{6}$$

with

$$\begin{aligned} a_1 &= b_1 = 45\,360(17n-50)\,, \\ a_2 &= 420(178n^2-951n+1250)\,, \\ a_3 &= 1290n^3-10\,849n^2+29\,100n-24\,500 \\ b_2 &= 420(178n^2-645n+350)\,, \\ b_3 &= 15n(86n^2-321n+190)\,. \end{aligned} \tag{7}$$

For $0 \le n \le 2.5$, it provides a good approximation over the whole radial extent of the star, because it yields fairly accurate values of the star radius, with a maximum error, at $n = 2.5$, of about 1.7%. The behavior for larger n was not discussed in [6], but one may foresee problems when approaching $n = 3$, because the coefficients a_1 and b_1 vanish for $n = 50/17 \simeq 2.941$. In fact, $y_n^{(2,2)}$ can be seen to exactly reduce to $y_n^{(1,1)}$ at $n = 50/17$, and we may consequently expect loss of accuracy on both sides of this value.

Things are a little worse, however, because of other changes of sign in the coefficients, which yield a complicated structure for the roots of the numerator of (6), $x_{1\pm}^2 = (-a_2 \pm \sqrt{a_2^2 - 4a_1a_3})/2a_3$.