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Effect of noise on extreme events probability in a one-dimensional nonlinear Schrödinger equation

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ABSTRACT

We report a numerical observation that multiplicative random forcing (noise) significantly increases the probability of formation of extreme events in the one-dimensional, focusing nonlinear Schrödinger equation. Furthermore, this phenomenon is sensitive to the noise's spatial correlation length. Highly correlated multiplicative noise may increase the probability of extreme events even when the average nonlinearity of the system is weak. On the contrary, noise with short spatial correlations substantially increases the probability of extreme events only for sufficiently strong average nonlinearity.

1. Introduction

Occurrence of waves whose amplitude exceeds the average by several standard deviations has been actively studied in such diverse areas as nonlinear optics and water waves; see, e.g., recent reviews [1–4] and references therein. Below we will refer to such waves as rogue waves irrespective of the physical context in which they occur. The nonlinear Schrödinger equation (NLS) has been considered as a toy model that is capable of producing rogue waves in fiber optics (see, e.g., [5]). In the oceanic wave theory, the NLS describes the initial stage of evolution of a weakly modulated wave packet. While this envelope equation cannot describe formation of a rogue wave, which is a single wave event, it can still indicate where a rogue wave may occur (see, e.g., [6–10] and references therein). Notably, the NLS model does predict a higher probability of observing rogue waves than the linear model.

In this letter we report a numerical observation that noise terms included in the NLS further increase the probability of rogue waves formation. Furthermore, certain types of noise do so considerably more than others, with a spatially highly correlated multiplicative noise resulting in the most prominent such an increase. We emphasize that since the NLS is only a toy model of rogue waves, then our stochastic modification of that equation is also meant only to exhibit a *trend* whereby a certain combination of nonlinearity, dispersion, and stochastic forcing results in a significant increase of the probability of rogue waves.

It should be noted that the stochastic NLS has also been extensively studied in diverse applications. For example, in fiber optical communications, additive noise has most often been used to model the effect of spontaneous emission from amplifiers, as the signal propagates in a transmission line (see, e.g., [11]), while multiplicative noise could model the effect of a fluctuating pump in a Raman amplifier (see, e.g., [12]). Most of those studies focused on how the noise affects a single soliton, although (wavelength-dependent) noise and damping were also considered in studies of wave turbulence (see, e.g., [13,14] and references therein).

The model that we consider here is

$$iu_t + \beta u_{xx} + \gamma |u|^2 u = -i\alpha u - i\varepsilon u_{xxxx} + \xi + \eta u, \qquad (1)$$

where β, γ are the dispersion and nonlinearity coefficients, α is a wavelength-independent damping coefficient, and the term εu_{xxxx} accounts for the energy loss at very high wavenumbers. In the context of water waves, such a term accounts for transfer of energy to very short waves (e.g., via white-capping) that are not resolved by the model (see, e.g., a related discussion in [15,16]). Its inclusion in the model does not qualitatively affect its predictions but does allow us to avoid using a very wide spectral domain and hence a very small time step in our numerical simulations. The additive and multiplicative noises, ξ and η, are assumed to be complex-valued, independent of each other, and having zero average and the following correlation functions:

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$$\langle \xi^*(x_1,t_1)\xi(x_2,t_2)\rangle = 2D(|x_1-x_2|)\delta(t_1-t_2), \tag{2a}$$

$$\langle \eta^*(x_1,t_1)\eta(x_2,t_2)\rangle = 2C(|x_1-x_2|)\delta(t_1-t_2), \tag{2b}$$

$$\langle \xi(x_1,t_1)\xi(x_2,t_2)\rangle = \langle \eta(x_1,t_1)\eta(x_2,t_2)\rangle = 0, \tag{2c}$$

where $\langle \ldots \rangle$ denotes averaging over an ensemble of noise realizations, the asterisk denotes a complex conjugate, and D and C are real-valued. In time, the noises are treated in the Stratonovich sense (see, e.g., Chap. 5 in [17]), whereby their correlation time is assumed to be much smaller than any other time scale in the model, but still finite.

In space, the noises must have finite (essentially nonzero) correlation lengths because, as we will show below, the spatial scale of the solution u is of the same order. A spatially correlated noise can be related to a spatially uncorrelated noise w, whose Fourier transform satisfies $\langle \hat{w}^*(k_1,t_1)\hat{w}(k_2,t_2)\rangle = 2\delta(k_1-k_2)\delta(t_1-t_2)$ and $\langle \hat{w}(k_1,t_1)\hat{w}(k_2,t_2)\rangle = 0$, via

$$\{\xi(x,t),\ \eta(x,t)\} = \int G_{\{\xi,\,\eta\}}(k)\hat{w}_{\{\xi,\,\eta\}}(k,t)e^{-ikx}dk; \tag{3a}$$

here $\{\ldots\}$ denotes grouping of terms. Then

$$\{D(x),\ C(x)\} = \int |G_{\{\xi,\,\eta\}}(k)|^2 e^{-ikx}dk, \tag{3b}$$

and the fact that D and C are real implies that $|G_{\{\xi,\,\eta\}}(k)|^2$ are symmetric. In what follows we will use notations

$$D(0) \equiv D_0, \qquad C(0) \equiv C_0. \tag{3c}$$

In (3a) and everywhere below, if the limits of integration are not indicated, they are assumed to be infinite.

Let us note that within the framework of the NLS as a model for oceanic waves, a combination of the multiplicative noise and damping terms, $(-\alpha+\eta)u$, can be interpreted as a result of combined action of the wavelength-independent damping and the forcing due to wind [18–20]. Here $(-\alpha)$ is the net damping rate due to both wavelength-independent loss mechanisms and the constant part of wind forcing, while η is attributed to the variable part of the forcing. In the oceanic waves context, an interpretation of the additive noise, ξ, is less clear. However, we include it due to both a formal reason explained in the next paragraph as well as for the generality of our toy model. It will follow from our results that it is the multiplicative noise term that is responsible for the main effect reported in this work.

One of our key assumptions is that we consider the evolution and, in particular, formation of rogue waves in the *statistically steady* state of model (1). (Below we will omit the modifier 'statistically' for brevity.) This implies that there must be a balance, on average, between the influx of energy to the system due to noise and the energy dissipation due to the α- and ε-terms. In linear systems, such a balance is well known as a form of the fluctuation-dissipation theorem, whereby the intensity of noise and the dissipation rate in a steady state must be related (see Eq. (6) below). Note that this balance for a nonzero solution u in (1) can only be achieved for a nonzero additive noise: if a multiplicative noise alone is present, then the solution will either blow up (due to a purely linear mechanism) or decay to zero. Thus, if one neglects the ε-term for a moment, the constants α, D_0, and C_0 must be related by (6) to guarantee the existence of a steady state. It may seem, and perhaps is, unphysical that the damping rate α, which is an intrinsic property of the wave model, and the noise intensities D_0 and C_0, which characterize the noises external to the model, are related. A more physical damping mechanism, at least for oceanic waves, may be one where energy is dissipated primarily in high wave numbers [13,14]. The reason why we consider the situation where almost all energy is lost due to the wavelength-independent damping is that in this case, it is possible to control the time-average nonlinearity using some analytical estimates. Such a control is required for a careful determination of sources that affect our main conclusion.

Let us note that since it is the noise that drives the model into the steady state, then the spatial spectral bandwidth of the steady-state solution must, on average, be on the order of (or greater than) that of the noise. This simply follows from the fact that terms in (1) must balance out. If the noise contribution to (1) is considerably less than that of the nonlinear term, then the spectrum of noise can be narrower than the spectrum of the average solution. However, it cannot be wider (in the order of magnitude sense); indeed, a wide-band noise would excite high wavenumbers in the steady-state solution, thereby widening its spectrum. In our simulations, we have observed the spectral bandwidths of the solution and the noise to be within a factor of two from one another, except for the moments where a rogue wave would form, at which point the solution's bandwidth would considerably exceed that of the noise. This is the physical reason why we consider only correlated noises, as per (2), in this work. This situation should be contrasted to that in optical communications: The noise bandwidth there considerably exceeds that of the useful signal, but the signal is *not* in statistical equilibrium with the noise in a telecommunication system.

The main part of this work is organized as follows. In Section 2 we justify the choice of some of the simulation parameters that guarantee that the numerical results are statistically significant (and yet do not require prohibitively long simulation times). The numerical results are reported in Section 3. In Section 4 we summarize our findings. Appendices A and B contain auxiliary derivations of the mass (a.k.a. number of particles) evolution and a brief description of the numerical method.

2. Estimating required simulation time

The probability of a rogue wave occurrence depends on the magnitude of the nonlinear term in (1) relative to the other terms. Therefore, to convincingly show that that probability is affected by noise as opposed to other factors, one must maintain a constant average $|u|^2$. This implies maintaining the ensemble average of the mass

$$N = \int_0^L |u|^2 dx, \tag{4}$$

where L is the length of the considered spatial domain. The evolution equation

$$d\langle N\rangle/dt = 2(D_0 L + C_0\langle N\rangle - \alpha\langle N\rangle) \tag{5}$$

can be derived using the method outlined in Appendix A; here we have ignored the action of the ε-term in (1) for reasons that were explained in the Introduction. Thus, in the statistically steady state,

$$\langle N\rangle_{\rm st} = D_0 L/(\alpha - C_0). \tag{6}$$

In order to have a specific value of $\langle N\rangle$ in a simulation, one must select values of D_0, C_0, and α satisfying (6). In this work we have always set the parameters so as to maintain the average mass of the solution near the value L:

$$\langle N\rangle \approx L \quad \Rightarrow \quad \frac{1}{L}\int_0^L \langle |u|^2\rangle\, dx \approx 1. \tag{7}$$

The next issue one needs to address is: In an individual observation (simulation), how much does N fluctuate around its aver-