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Unified scheme for correlations using linear relative entropy

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ABSTRACT

A linearized variant of relative entropy is used to quantify in a unified scheme the different kinds of correlations in a bipartite quantum system. As illustration, we consider a two-qubit state with parity and exchange symmetries for which we determine the total, classical and quantum correlations. We also give the explicit expressions of its closest product state, closest classical state and the corresponding closest product state. A closed additive relation, involving the various correlations quantified by linear relative entropy, is derived.

1. Introduction

Quantum entanglement in quantum systems, comprising two or more parts, constitutes a key concept to distinguish between quantum and classical correlations and subsequently to understand quantum–classical boundary. Besides its fundamental aspects, entanglement is commonly accepted of paramount importance in the development of quantum information science [1–6]. In fact, entangled states have found various applications in quantum information processing protocols as for instance quantum cryptography [7], quantum teleportation [8], quantum dense coding [9]. Nowadays, entanglement is recognized as a valuable resource in several communication and computational tasks [10–12]. In view of these remarkable realizations and implementations, the concept of entanglement is expected to have many other implications and applications in other areas of research, especially condensed matter physics.

Therefore, the quantification and the characterization of quantum correlations between the components of a composite quantum system have attracted a special attention during the last two decades. The experimental and theoretical efforts, deployed in this context, are essential to develop the appropriate strategies to prevent against the decoherence effects induced by the system-environment coupling (see for instance the recent works [13–15] and references therein). Different measures were introduced from different perspectives and for various purposes [16–22]. Probably the most familiar among them is the quantum discord [23,24] which goes beyond the entanglement of formation [25,26]. It is given by the difference of total and classical correlations existing in a bipartite system. Now, it is well understood that almost all quantum states, including unentangled (separable) ones, possess quantum correlations. However, the analytical evaluation of quantum discord requires extremization procedures that can be tedious to achieve [27–34]. To overcome this difficulty, a geometrical approach was proposed in [35]. It is based on the Hilbert–Schmidt norm in the space of density matrices. This measure provides explicit analytical expressions for pairwise quantum correlations. Clearly, Hilbert–Schmidt norm is not the unique distance which can be defined in the space of quantum states. Several distances are possible (trace distance, Bures distance, etc.) with their own advantages and drawbacks and each one might be useful for some appropriate purpose [36–39].

The states of any multipartite quantum system can be classified as being classical, quantum–classical and quantum states. Subsequently, the correlations can also be categorized in total, quantum, semi-classical (related to quantum–classical states) and classical correlations. This classification requires a specific measure (entropic or geometric distance) to decide about the dissimilarity between a given quantum state and its closest one without the desired property and to provide a consistent scheme to treat equally the different correlations. In this sense, using the relative entropy, an approach unifying the correlations in multipartite systems was recently developed in [40]. In particular, a very significant and interesting additivity relation was reported ($D + C = T + L$). It states

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that the sum of quantum D and classical C correlations is equal to the sum of total mutual correlations T and another quantity L that is exactly the difference between D and the quantum discord as originally introduced in [23,24].

However, it must be noticed that, despite its theoretical information meaning, the relative entropy is not symmetric in its arguments and therefore cannot be considered as a true metric distance. In the other hand, from an analytical point of view, the derivation of closed expressions of relative entropy based measures involves optimization procedures that are in general very complicated to perform. In this respect, a purely geometrical unified framework to classify the correlations in a given quantum state was discussed in [41,42]. Using the Hilbert–Schmidt norm and paralleling the definition of the geometric discord, the geometric measures of total and classical correlations in a two qubit system were derived in [41,42]. In contrast with the relative entropy, the additivity relation of the type $(D + C = T + L)$ is not, in general, satisfied.

In this paper, we introduce a linearized variant of relative entropy. We obtain the explicit analytical expressions of quantum and classical correlations in a two qubit system. The relation with the geometric measure based on Hilbert–Schmidt norm is established. We show that the linear relative entropy provides us with a simple approach to treat the different kinds of bipartite correlations in a common framework. This approach can be seen, in some sense, interpolating between the relative entropy-based [23,24] and Hilbert–Schmidt-based [41,42] classification schemes. More specifically, it provides us with a very simple way to perform the optimizations required in deriving closest product, classical and classical product states. We also show that the correlations satisfy a closed additivity.

This paper is organized as follows. In Section 2, we decompose the linear entropy in symmetric and anti-symmetric parts. We show that the antisymmetric part is related to quantum Jensen–Shannon divergence and the symmetric part is exactly the Hilbert–Schmidt distance [41]. Using the linear relative entropy, we obtain a closed additivity relation of the various bipartite correlations existing in a two qubit system. A comparison with Hilbert–Schmidt based approach is also investigated. As illustration, we consider, in Section 3, a bipartite system possessing the parity symmetry and invariant under qubits permutation. In this situation, the explicit derivations of the suitable closest product and classical states is achieved. The analytical expressions of total, quantum and classical correlations are obtained and the additivity relation is discussed. Concluding remarks close this paper.

2. Correlation quantifiers based on symmetrized linear relative entropy

2.1. Correlation quantifiers based on relative entropy

The relative entropy offers the appropriate scheme to unify the different kinds of correlations existing in multipartite systems [40]. It is the quantum analogue of the Kullback–Leibler divergence between two classical probability distributions and characterizes the dissimilarity between two quantum states. The relative entropy defined by

$$S(\rho\|\sigma) = -\operatorname{Tr}(\rho\log\sigma) - S(\rho), \tag{1}$$

constitutes a quantitative tool to distinguish between the states of a given degree of quantumness and gives the distance between them according to the nature of their properties ($S(\rho) = -\operatorname{Tr}(\rho\log\rho)$ is the von Neumann entropy). For a bipartite system, the total correlation $T = S(\rho\|\pi_\rho)$ is quantified by the relative entropy between a state ρ and its closest product state $\pi_\rho = \rho_A\otimes\rho_B$, where ρ_A and ρ_B denote the reduced density matrices of the subsystems. It writes as the difference of the von Neumann entropies [40]

$$T = S(\rho\|\pi_\rho) = S(\pi_\rho) - S(\rho). \tag{2}$$

Similarly, the quantum discord, which encompasses quantum correlations, is measured as the minimal distance between the state ρ and its closest classical state

$$\chi_\rho = \sum_{i,j} p_{i,j}|i\rangle\langle i|\otimes|j\rangle\langle j|, \tag{3}$$

where $p_{i,j}$ are the probabilities and $\{|i\rangle, |j\rangle\}$ local basis. It writes also as the difference between the von Neumann entropies of the states ρ and χ_ρ [40]

$$D = S(\rho\|\chi_\rho) = S(\chi_\rho) - S(\rho). \tag{4}$$

The classical correlation gives the distance between the closest classical state χ_ρ and its closest classical product state π_{χ_ρ}. It coincides with the difference of von Neumann entropies of the relevant states

$$C = S(\chi_\rho\|\pi_{\chi_\rho}) = S(\pi_{\chi_\rho}) - S(\chi_\rho). \tag{5}$$

In this approach the relative entropy-based quantum correlations or quantum discord D (4) does not coincide with the original definition of discord introduced in [23,24]. The difference is given by [40]

$$L = S(\pi_\rho\|\pi_{\chi_\rho}) = S(\pi_{\chi_\rho}) - S(\pi_\rho). \tag{6}$$

The entropy-based correlations T, D, C and L are expressed as differences of von Neumann entropies (Eqs. (2), (4), (5) and (6)) and they satisfy the following remarkable additivity relation [40]

$$T - D - C + L = 0. \tag{7}$$

It must be emphasized that the relative entropy (1) is not symmetric under the exchange $\rho\leftrightarrow\sigma$. In this respect, it cannot define a distance from a purely mathematical point of view. Moreover, as mentioned above, the analytical evaluation of relative entropy-based correlations requires intractable minimization procedures. To avoid this problem, the linear relative entropy offers an alternative way to get computable expressions of correlations existing in multipartite systems [41].

2.2. Linear relative entropy

The linear entropy

$$S_2(\rho) \doteq 1 - \operatorname{Tr}(\rho^2)$$

is related to the degree of purity, $P = \operatorname{Tr}(\rho^2)$, and therefore reflects the mixedness in the state ρ. It is defined as a linearized variant of von Neumann entropy by approximating $\log\rho$ by $\rho - \mathbb{I}$ where $\mathbb{I}$ stands for the identity matrix. Accordingly, the relative entropy (1) can be linearized as follows [41]

$$S_l(\rho_1\|\rho_2) = \operatorname{Tr}\rho_1(\rho_1 - \rho_2). \tag{8}$$

It is not symmetric under the interchange of the states ρ_1 and ρ_2. To define a symmetrized linear relative entropy, $S_l(\rho_1\|\rho_2)$ is decomposed as the sum of two terms: symmetric and antisymmetric. The symmetric part is defined by

$$S_+(\rho_1\|\rho_2) = S_l(\rho_1\|\rho_2) + S_l(\rho_2\|\rho_1). \tag{9}$$

The antisymmetric term is given by