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Hydrogen atom in rotationally invariant noncommutative space

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ABSTRACT

We consider the noncommutative algebra which is rotationally invariant. The hydrogen atom is studied in a rotationally invariant noncommutative space. We find the corrections to the energy levels of the hydrogen atom up to the second order in the parameter of noncommutativity. The upper bound of the parameter of noncommutativity is estimated on the basis of the experimental results for $1s$–$2s$ transition frequency.

1. Introduction

The idea that space might have a noncommutative structure was proposed by Heisenberg and was formalized by Snyder [1]. In recent years, noncommutativity has met considerable interest due to development of String Theory [2,3] and Quantum Gravity [4].

In the canonical version of noncommutative space the coordinate and momentum operators satisfy the following commutation relations

$$[X_i, X_j] = i\hbar\theta_{ij}, \quad (1)$$

$$[X_i, P_j] = i\hbar\delta_{i,j}, \quad (2)$$

$$[P_i, P_j] = 0, \quad (3)$$

where θ_{ij} is a constant antisymmetric matrix. Many physical problems have been considered in this space (see, for instance, [5] and references therein). Among them the hydrogen atom was studied [6–13]. In [6] the corrections to the energy levels of hydrogen atom were found up to the first order in the parameter of noncommutativity. In this article the authors also obtained the corrections to the Lamb shift within the noncommutative quantum electrodynamic theory. In [7] the hydrogen atom was studied as a two-particle system in the case when particles of opposite charges feel opposite noncommutativity. The quadratic Stark effect was examined in [9]. New result for shifts in the spectrum of hydrogen atom in noncommutative space was presented in [10]. In [11] the hydrogen atom energy levels were calculated in the framework of the noncommutative Klein-Gordon equation. In [12,13] the Dirac equation with a Coulomb field was considered in noncommutative space.

Hydrogen atom problem was also considered in the case of space–time noncommutativity [14–18], space–space and momentum–momentum noncommutativity [19–21]. Full phase-space noncommutativity in the Dirac equation was considered in [22]. The authors concluded that in order to preserve gauge symmetry one should discard configuration space noncommutativity. In this article the hydrogen atom was studied. It was shown that only hyperfine structure is affected by the momentum noncommutativity. In the case of configuration space noncommutativity the issue of the gauge invariance is more complicated and was studied, for example, in [23]. In this article the authors prove the existence of generalized gauge transformations.

It is worth noting that in a three-dimensional noncommutative space we face the problem of the rotational symmetry breaking [6, 24]. In order to preserve this symmetry new classes of noncommutative algebras were explored. For instance, in [25] the rotational invariance was preserved by foliating the space with concentric fuzzy spheres. In [26] the rotationally symmetric noncommutative space was constructed as a sequence of fuzzy spheres. In this case the exact solution of the hydrogen atom problem was found. In [27] the curved noncommutative space was introduced to maintain the rotational symmetry and the hydrogen atom spectrum was studied.

Also in order to preserve the rotation invariance the promotion of the parameter of noncommutativity to an operator in Hilbert space was suggested and the canonical conjugate momentum of

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this operator was introduced [28]. In this case the author considered an isotropic harmonic oscillator. The way to maintain the N dimensional rotation invariance was considered in [29]. In this article the coordinates are represented by operators. The measured values of these operators are expectations between generalized coherent states. In [30] the noncommutative coordinates invariant under rotations were presented. The author suggested identification of the coordinates with the boost operators in SO(1, 3).

Note, however, that in a two-dimensional space the rotational symmetry survives even in the canonical version of noncommutativity $[X, Y] = i\hbar\theta$, where θ is a constant.

Much attention has also been received to studying of spherically symmetric noncommutative spaces [31–33], considering of the problem of violation of the Lorentz invariance (see for example [34–36]). For instance, noncommutative gauge theory without Lorentz violation was proposed in [34]. The authors considered a new class of noncommutative theories in which the parameter of noncommutativity is promoted to an antisymmetric tensor that transforms as a Lorentz tensor. The generalizations of the operator trace and star product were found. Therefore, the Lagrangian was considered as a θ-integrated quantity.

In recent years, a new class of noncommutative theories that involve additional degrees of freedom namely spin degrees of freedom has been considered (see, for example, [37,38] and references therein). An important advantage of these theories is rotational invariance of noncommutative algebras. It is worth mentioning that in the case of spin-1/2 noncommutativity one is forced to extend the wave function to a two-component wave function.

In this article we consider the way of constructing the rotationally invariant noncommutative space by the generalization of the constant matrix θ_{ij} to a tensor. For this purpose we consider the idea to involve additional degrees of freedom. But in contrast to the spin noncommutativity we propose to involve some additional space coordinates. Namely, we propose to construct the tensor of noncommutativity with the help of additional coordinates governed by the harmonic oscillators. The corresponding algebra is rotationally invariant. Extension of noncommutative algebra with the help of additional space coordinates allows us to describe quantum system by one-component wave function. The hydrogen atom is considered in the rotationally invariant noncommutative space. We study the perturbation of the hydrogen atom energy levels caused by the noncommutativity of coordinates.

The article is organized as follows. In Section 2 the way to preserve the rotational symmetry is considered. In Section 3 the Hamiltonian of the hydrogen atom is studied in the rotationally invariant noncommutative space. The corrections to the energy levels of the hydrogen atom are found up to the second order in the parameter of noncommutativity in Section 4. In addition, the way to construct an effective Hamiltonian which is rotationally invariant is proposed. Section 5 is devoted to calculation of the corrections to the ns levels. The upper bound of the parameter of noncommutativity is estimated in Section 6. Conclusions are presented in Section 7.

2. Rotationally invariant noncommutative space

In order to preserve the rotational symmetry we propose to define the tensor of noncommutativity as follows

$$\theta_{ij} = \frac{\alpha}{\hbar}(a_i b_j - a_j b_i), \tag{4}$$

where α is a dimensionless constant and a_i, b_i are governed by a rotationally symmetric system. We suppose for simplicity that a_i, b_i are governed by the harmonic oscillator

$$H_{osc} = \frac{(p^a)^2}{2m} + \frac{(p^b)^2}{2m} + \frac{m\omega^2 a^2}{2} + \frac{m\omega^2 b^2}{2}. \tag{5}$$

Note that harmonic oscillator has two independent units of measurement, namely the unit of length $\sqrt{\frac{\hbar}{m\omega}}$ and the unit of energy $\hbar\omega$. It is generally believed that the parameter of noncommutativity of coordinates is of the order of the Planck scale, therefore we put

$$\sqrt{\frac{\hbar}{m\omega}} = l_p, \tag{6}$$

where l_p is the Planck length. Independently of (6) we also consider the limit $\omega \to \infty$. In this case the distance between the energy levels of harmonic oscillator tends to infinity. Therefore, harmonic oscillator put into the ground state remains in it.

So, we propose to consider the following commutation relations

$$[X_i, X_j] = i\alpha(a_i b_j - a_j b_i), \tag{7}$$

$$[X_i, P_j] = i\hbar\delta_{i,j}, \tag{8}$$

$$[P_i, P_j] = 0. \tag{9}$$

The coordinates a_i, b_i, and momenta p^a_i, p^b_i satisfy the ordinary commutation relations $[a_i, a_j] = 0$, $[a_i, p^a_j] = i\hbar\delta_{i,j}$, $[b_i, b_j] = 0$, $[b_i, p^b_j] = i\hbar\delta_{i,j}$, also $[a_i, b_j] = [a_i, p^b_j] = [b_i, p^a_j] = [p^a_i, p^b_j] = 0$. It is worth noting that a_i, b_i commute with X_i and P_i and therefore θ_{ij} given by (4) commutes with X_i and P_i too. So, X_i, P_i and θ_{ij} satisfy the same commutation relations as in the case of the canonical version of noncommutativity. Therefore, in this sense algebra (7)–(9) is equivalent to (1)–(3). Note that in the case of the spin noncommutativity the situation is different. Namely, the tensor of noncommutativity is connected with the spin variable and does not commute with coordinates (see for instance [37]).

It is convenient to use the following representation

$$X_i = x_i - \frac{\alpha}{2\hbar}\sum_j (a_i b_j - a_j b_i) p_j, \tag{10}$$

$$P_i = p_i, \tag{11}$$

where the coordinates x_i and momenta p_i satisfy the ordinary commutation relations $[x_i, x_j] = 0$, $[x_i, p_j] = i\hbar\delta_{i,j}$. Taking into account (10), it is clear that

$$\left[X_i, p^a_j\right] = \frac{i\alpha}{2}\left(b_i p_j - \delta_{i,j}(\mathbf{b}\cdot\mathbf{p})\right), \tag{12}$$

$$\left[X_i, p^b_j\right] = -\frac{i\alpha}{2}\left(a_i p_j - \delta_{i,j}(\mathbf{a}\cdot\mathbf{p})\right), \tag{13}$$

$$\left[P_i, p^a_j\right] = \left[P_i, p^b_j\right] = 0. \tag{14}$$

It is easy to show that noncommutative coordinates can be represented in a more convenient form

$$X_i = x_i + \frac{1}{2}[\boldsymbol{\theta} \times \mathbf{p}]_i, \tag{15}$$

where

$$\boldsymbol{\theta} = \frac{\alpha}{\hbar}[\mathbf{a} \times \mathbf{b}]. \tag{16}$$

So, we can represent noncommutative coordinates X_i and momenta P_i by the coordinates x_i and momenta p_i which satisfy the ordinary commutation relations. Therefore the Jacobi identity is satisfied and this can be easily checked for all possible triplets of operators.

Algebra (7)–(9) is manifestly rotationally invariant. It is clear that commutation relation (7) remains the same after rotation $X'_i = U(\varphi)X_i U^+(\varphi)$, $a'_i = U(\varphi)a_i U^+(\varphi)$, $b'_i = U(\varphi)b_i U^+(\varphi)$

$$\left[X'_i, X'_j\right] = i\alpha\left(a'_i b'_j - a'_j b'_i\right), \tag{17}$$