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Local nonlinear electrodynamics

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ABSTRACT

The propagation of electromagnetic waves in dielectric media characterized by the coefficients $\varepsilon^{\alpha}{}_{\beta} = \varepsilon^{\alpha}{}_{\beta}(E^{\mu}, B^{\mu}, \partial_{\nu}E^{\mu}, \partial_{\nu}B^{\mu})$ and $\mu^{\alpha}{}_{\beta} = \mu^{\alpha}{}_{\beta}(E^{\mu}, B^{\mu}, \partial_{\nu}E^{\mu}, \partial_{\nu}B^{\mu})$ is examined in the eikonal approximation of electrodynamics. Employing the techniques Hadamard–Papapetrou (HPD) and Spacetime Integration (STI), we derive the dispersion relation, the polarization modes and effective geometry associated to the model.

1. Introduction

The propagation of electromagnetic waves in nonlinear theories of electromagnetism has, recently, attracted great interest in the scientific community. This problem can be investigated under two different aspects. We can examine this question in the regime of intense fields [1–3], and also in the context of material media [4–15]. In both cases, the field equations that govern the electromagnetic phenomena are nonlinear. In the regime of intense fields, the theory is analytically built from a nonlinear lagrangean, which generally is a function of both the Lorentz invariants $F := F^{\mu\nu}F_{\mu\nu}$ and $G := F^{\mu\nu}F^{*}_{\mu\nu}$ of the electromagnetic field [16]. In the context of materials media, the Maxwell equations must be supplemented with constitutive relations

$$D^{\alpha} = \varepsilon^{\alpha}{}_{\beta}E^{\beta},$$
$$H^{\alpha} = \mu^{\alpha}{}_{\beta}B^{\beta}, \tag{1}$$

where the coefficients $\varepsilon^{\alpha}{}_{\beta}$ and $\mu^{\alpha}{}_{\beta}$ represent the dielectric matrices. They are usually called electric permittivity and magnetic permeability, respectively. All information about the dielectric properties of the medium can be obtained from the constitutive relations [17–20]. In this case, the structure of the propagation of waves is dependent on the behavior of the medium acted upon by the external fields as encoded in terms of certain functions which in general are nonlinear. The electrodynamics in material media is also considered as a possible scenario for the investigation of analog models for gravitational phenomena [21].

This work aims to obtain and discuss the solutions to the Fresnel eigenvalue equation which describes the propagation of light rays in material media whose dielectric matrices have a functional dependence on the electromagnetic field and its first order derivatives. The analysis is restricted to local electrodynamics, where dispersive effects are neglected. Only monochromatic waves are considered, thus avoiding ambiguities with respect to the velocity of the waves.

Section 2 presents two alternative techniques of propagation of field discontinuities: (i) Hadamard–Papapetrou (HPD), and (ii) Spacetime Integration (STI). Section 3 presents the eigenvalue equation for our model. Section 4 obtains the dispersion relation and the permitted polarization modes, as well as the optical metric for the model are analyzed. Section 5 summarizes our results.

A Minkowskian spacetime described in an adapted Cartesian coordinate system is used throughout this work. The units are such that $c = 1$. The spacetime metric is denoted by $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. For an arbitrarily given function $F(x^{\mu})$, its partial derivatives $\partial_{\mu}F$ with respect to any given spacetime coordinate x^{μ} is denoted by $F_{,\mu}$. All quantities are referred to as measured by the geodetic observer $V^{\mu} = \delta^{\mu}_{0}$, where δ^{μ}_{ν} denotes the Kronecker tensor and $h^{\mu}_{\nu} := \delta^{\mu}_{\nu} - V^{\mu}V_{\nu}$ is the projector onto the three-dimensional rest space of this observer V^{μ}. For any given pair of space-like vectors $X^{\mu} = (0, \vec{X})$ and $Y^{\mu} = (0, \vec{Y})$, we have $(\vec{X} \cdot \vec{Y}) = -X^{\mu}Y_{\mu}$.

2. The formalism of shock waves

2.1. Hadamard–Papapetrou (HPD)

The technique used in the HPD formalism is to analyze the discontinuity of a function F by an orientable hypersurface in a

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differentiable manifold M [22,23]. In this work, we are interested in analyzing the discontinuity of a function F across a space-like (or possibly light-like) orientable borderless hypersurface. Let Σ be such a hypersurface, defined in terms of a given function Φ by Σ: $\Phi(x^\mu) = 0$. Such Σ then splits the spacetime into three consistently defined sets [4]. Let X^- be the union of sets of spacetime points P^- in the past of P for each $P \in \Sigma$, and similarly X^+ be the union of sets of spacetime points P^+ in the future of P for each $P \in \Sigma$. Causality of spacetime ensures that X^+ and X^- are two disjoint sets. For each point $P_0 \in \Sigma$, any sufficiently small neighborhood U_{P_0} of P_0 is partitioned into three disjoint regions: $U^-_{P_0} \subset X^-$, $U^+_{P_0} \subset X^+$ and $U^0_{P_0} \subset \Sigma$. Let r be the radius of this neighborhood U_{P_0}, and let also $P^- \in U^-_{P_0}$ and $P^+ \in U^+_{P_0}$ be any two neighbor points from P_0 arbitrarily chosen on opposite sides of Σ. Consider an arbitrary function $F(x^\mu)$ (or tensor field of arbitrary rank, whose indices being omitted for convenience of notation) defined in U_{P_0}. The discontinuity of $F(x^\mu)$ across Σ is then defined as

$$[F]_\Sigma(P_0) := \lim_{r\to 0^+}\big[F\big(P^+\big) - F\big(P^-\big)\big]. \tag{2}$$

Suppose that the function $F(x^\mu)$ has a vanishing discontinuity across Σ, *i.e.* that $[F]_\Sigma(P) = 0$ at each point $P \in \Sigma$. Papapetrou has shown [23] that the discontinuity of $F_{,\mu}$ across Σ has the form

$$[F_{,\mu}]_\Sigma(P) = \tilde{F} K_\mu, \tag{3}$$

where $\tilde{F}$ is a function defined on Σ, with the same rank and with the same algebraic symmetries of F, while K_μ is the vector normal to Σ defined by $K_\mu := \partial_\mu \Phi$. More generally, if the function $F(x^\mu)$ is such that all its derivatives $F_{,\mu_1\mu_2\ldots\mu_i}$ from order zero to order i have null discontinuities in Σ, then its derivative $F_{,\mu_1\mu_2\ldots\mu_i\mu_{i+1}}$ of order $(i+1)$ presents discontinuity through Σ as

$$[F_{,\mu_1\mu_2\ldots\mu_i\mu_{i+1}}]_\Sigma = \tilde{F} K_{\mu_1} K_{\mu_2} \ldots K_{\mu_i} K_{\mu_{i+1}}, \tag{4}$$

where $\tilde{F}$ is a function defined on Σ, with the same rank and with the same algebraic symmetries of F.

2.2. Spacetime Integration (STI)

Maxwell equations constitute a system of coupled partial differential equations of first order, and must hold in the vicinity of any given point on which the fields are continuous. However, if there is any domain of spacetime for which the fields are discontinuous, then in this domain these equations are not well defined. Thus, it becomes necessary to replace this set of differential equations by a new set of equations to give us information about the discontinuity of the fields. Consider an open and connected limited domain G of the spacetime, and let Γ: $\phi(x^\mu) = 0$ be a hypersurface which is the boundary of G. Admit that Γ is continuous and orientable, and that it has continuous by parts tangent hyperplanes. Let also a function f be continuous in $\bar{G} = G \cup \Gamma$ and with continuous partial derivatives in G. Then, it holds the identity (Stokes's Theorem)

$$\int_G (\partial_\mu f)\, d\Omega = \int_\Gamma f n_\mu \, dS, \tag{5}$$

where $\lambda := \pm(\partial_\mu \phi \partial^\mu \phi)^{-1/2}$ and $n_\mu := \lambda \partial_\mu \phi$, while $d\Omega$ is an infinitesimal element of domain G and dS is an infinitesimal element of hipersurface Γ. Therefore, n_μ is a normalized vector which is orthogonal to the hypersurface Γ. The signal of λ is chosen to ensure that the vector n_μ is directed outwards from G.

Consider a domain G of the spacetime over which the fields $\vec{E}$ and $\vec{B}$, the inductions $\vec{D}$ and $\vec{H}$, and the sources ρ and $\vec{J}$, are all continuous and have continuous partial derivatives. Integrating Maxwell equations

$$\partial_\mu D^\mu = \rho, \tag{6}$$
$$V^\nu \partial_\nu D^\mu + \eta^{\mu\nu\alpha\beta} V_\alpha \partial_\nu H_\beta = J^\mu, \tag{7}$$
$$\partial_\mu B^\mu = 0, \tag{8}$$
$$V^\nu \partial_\nu B^\mu - \eta^{\mu\nu\alpha\beta} V_\alpha \partial_\nu E_\beta = 0, \tag{9}$$

over such G and using the identity equation (5), we have

$$\begin{aligned}
&\int_\Gamma \big\{\vec{n}\times\vec{H} - \vec{D}\lambda(\partial_t\phi)\big\}\, dS = \int_G \vec{J}\, d\Omega,\\
&\int_\Gamma \big\{\vec{n}\times\vec{E} + \vec{B}\lambda(\partial_t\phi)\big\}\, dS = 0,\\
&\int_\Gamma \vec{n}\cdot\vec{D}\, dS = \int_G \rho\, d\Omega,\\
&\int_\Gamma \vec{n}\cdot\vec{B}\, dS = 0,
\end{aligned} \tag{10}$$

where $\vec{n} := \lambda\vec{\nabla}\phi$. If the vector fields ρ, $\vec{J}$, $\vec{E}$, $\vec{B}$, $\vec{D}$ and $\vec{H}$ are all continuous and have continuous partial derivatives in G, then Eqs. (10) are fully equivalent to Maxwell equations (6)–(9). However, Eqs. (10) hold also for discontinuous fields. We can therefore consider these last equations as a generalization of Maxwell equations. Discontinuous fields that solve these integral equations are called *weak solutions* of Maxwell equations [24,25].

Assume that Γ_0: $\Phi(x^\mu) = 0$ is a regular orientable hypersurface where the electromagnetic fields are discontinuous. This hypersurface Γ_0 is a continuously differentiable submanifold that cuts an open connected limited domain G of spacetime in two open and disjoint subdomains G_1 and G_2. The subdomain G_1 is enclosed by Γ_1 and Γ_0, while the subdomain G_2 is enclosed by Γ_2 and Γ_0. By means of a procedure similar to the one which leads to Eqs. (10), one can obtain two new sets of integral equations valid in subdomains G_1 and G_2 directly from Maxwell equations. The compatibility of these three sets of integral equations yields

$$\begin{aligned}
&\vec{\nabla}\Phi \times [\vec{H}]_{\Gamma_0} - (\partial_t\Phi)[\vec{D}]_{\Gamma_0} = 0,\\
&\vec{\nabla}\Phi \times [\vec{E}]_{\Gamma_0} + (\partial_t\Phi)[\vec{B}]_{\Gamma_0} = 0,\\
&\vec{\nabla}\Phi \cdot [\vec{D}]_{\Gamma_0} = 0,\\
&\vec{\nabla}\Phi \cdot [\vec{B}]_{\Gamma_0} = 0,
\end{aligned} \tag{11}$$

where $[\vec{X}]_{\Gamma_0}$ stands for the discontinuity of the quantity $\vec{X}$ across Γ_0 making use of the notation of Eq. (2). Such a set of equations can be formally obtained from Maxwell equations (6)–(9) by replacing the differential operators ∂_μ by the multiplicative operators $\partial_\mu \Phi$.

3. Fresnel equation

Various models of nonlinear electrodynamics where the dielectric matrices $\varepsilon^\alpha{}_\beta$ and $\mu^\alpha{}_\beta$ have a given functional dependence with respect to external electromagnetic fields applied to the medium are well known in the literature. We consider a model where such dielectric matrices depend not only on the external electromagnetic fields E^μ and B^μ, but also on their spatial derivatives. Accordingly, our analysis corresponds to local nonlinear electrodynamics.

Consider the Maxwell equations (6)–(9) and the constitutive relations (1) of a given material medium is such that its dielectric parameters $\varepsilon^\alpha{}_\beta$ and $\mu^\alpha{}_\beta$ have the functional form