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Fan sub-equation method for Wick-type stochastic partial differential equations

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ABSTRACT

An improved algorithm is devised for using Fan sub-equation method to solve Wick-type stochastic partial differential equations. Applying the improved algorithm to the Wick-type generalized stochastic KdV equation, we obtain more general Jacobi and Weierstrass elliptic function solutions, hyperbolic and trigonometric function solutions, exponential function solutions and rational solutions.

1. Introduction

Korteweg–de Vries (KdV) equation is one of the most popular soliton equations. To the best of our knowledge it is Wadati [1] who firstly introduced and studied stochastic KdV equation, and shown the diffusion of solitons of the KdV equation under Gaussian noise. Wadati and Akutsu [2] also studied the behaviors of solitons under the Gaussian white noise of the stochastic KdV equations with and without damping. In addition, Wadati [3] presented a nonlinear partial differential equation (PDE) describing wave propagation in random media. Recently, the study of the random waves have attached much attention and gradually becomes an important subject of stochastic partial differential equations (SPDEs). Many authors, such as de Bouard and Debussche [4,5], Debussche and Printems [6,7], and Konotop and Vázques [8], have studied the stochastic KdV equation. Holden et al. [9] gave white noise functional approach to research SPDEs in Wick versions. Based on the work in [9], some analytic methods for nonlinear PDEs are extended to solve Wick-type SPDEs. We suggest readers to see the remarkable work in [10–24].

In [25–29], Fan et al. proposed and developed an algebraic method which greatly exceeds the applicability of the existing tanh, extended tanh methods and Jacobi elliptic function expansion method to construct in a uniform way a series of exact solutions of nonlinear PDEs including polynomial solutions, exponential solutions, rational solutions, trigonometric periodic wave solutions, hyperbolic and solitary wave solutions and Jacobi and Weierstrass doubly periodic wave solutions. This method was later improved in different manners [30–35]. More recently, Wang and Wang did interesting work to extend Fan sub-equation method [25–29] to the Wick-type $(2+1)$-dimensional stochastic Borer–Kaup equation and obtained more general formal solutions.

The present Letter is motivated by the desire to devise an improved algorithm for using Fan sub-equation method [25–29] to construct new and more general exact solutions of nonlinear SPDEs. In order to illustrate the validity and advantages of the improved algorithm, we will apply it to a generalized stochastic KdV equation in the form [21]

$$U_t + 2H_2(t) \diamond U + \left[H_1(t) + H_2(t)x\right] \diamond U_x - 3cH_3(t) \diamond U \diamond U_x + H_3(t) \diamond U_{xxx} = 0, \tag{1}$$

where $H_i(t)$ $(i = 1, 2, 3)$ are the white noise functions, and $\diamond$ is the Wick product on the Hida distribution space $S^*(\mathbb{R}^d)$ (see [9,10] for more details).

The rest of this Letter is organized as follows. In Section 2, we describe the improved algorithm for using Fan sub-equation method to solve nonlinear SPDEs. In Section 3, we use this improved algorithm to solve the generalized stochastic KdV equation (1). In Section 4, some conclusions and discussions are given.

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2. Description of Fan sub-equation method for Wick-type SPDEs

We outline in this section the main steps of Fan sub-equation method for solving Wick-type SPDEs.

Step 1. With the aid of the Hermite transformation, we transform the Wick-type equation

$$A^{\diamond}(t, x, \partial_t, \nabla_x, U, \omega) = 0 \tag{2}$$

into an ordinary products equation (variable-coefficient PDE)

$$\widetilde{A}(t, x, \partial_t, \nabla_x, \widetilde{U}, z_1, z_2, \ldots) = 0, \tag{3}$$

where $U = U(t, x, \omega)$ is the unknown (generalized) stochastic process, $\widetilde{U} = \mathcal{H}(U)$ is the Hermite transformation of U, $z_1, z_2, \ldots$ are complex numbers, and the operators $\partial_t = \frac{\partial}{\partial t}$, $\nabla_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d})$ when $x = (x_1, x_2, \ldots, x_d)$.

Step 2. Denoting $\widetilde{U}(t, x, z) = u(t, x, z)$ and supposing Eq. (3) has solution in the more general form [33]

$$u(t, x, z) = a_0 + \sum_{i=1}^{n} \{a_i \phi^i(\xi) + b_i \phi^{-i}(\xi) + c_i \phi^{i-1}(\xi)\phi'(\xi) + d_i \phi^{-i}(\xi)\phi'(\xi)\}, \tag{4}$$

for each $z = (z_1, z_2, \ldots) \in \mathbb{K}_q(r) = \{z \in \mathbb{C}^{\mathbb{N}}$ and $\sum_{\alpha \neq 0} |z^\alpha|^2 (2\mathbb{N})^{q\alpha} < r^2\}$ for some q, r, here a_0, a_i, b_i, c_i, d_i and ξ are all functions of (x, z, t) to be determined later, $\phi'(\xi) = \mathrm{d}\phi(\xi)/\mathrm{d}\xi$, and $\phi(\xi)$ satisfies Fan's subsidiary ordinary differential equation with parameters $h_\rho(\rho = 0, 1, 2, \ldots, r)$

$$\phi'^2(\xi) = \sum_{\rho=0}^{r} h_\rho \phi^\rho(\xi), \tag{5}$$

and hence

$$\phi''(\xi) = \frac{1}{2}\sum_{\rho=1}^{r} \rho h_\rho \phi^{\rho-1}(\xi), \qquad \phi'''(\xi) = \frac{1}{2}\sum_{\rho=2}^{r} \rho(\rho-1) h_\rho \phi^{\rho-2}(\xi)\phi'(\xi), \ldots. \tag{6}$$

Step 3. Balancing the highest order partial derivative term with the nonlinear term(s) in Eq. (3), we get a restriction between n and r, form which the proper values of n and r can be determined.

Step 4. Substituting Eq. (4) along with Eqs. (5) and (6) given the value of n and r respectively into Eq. (3), collecting coefficients of $x^p \phi^j(\xi)\phi'^q(\xi)$ $(p, q = 0, 1;\ j = 0, \pm 1, \pm 2, \ldots)$, then setting each coefficient to zero, we derive a set of over-determined PDEs for a_0, a_i, b_i, c_i, d_i and ξ.

Step 5. Solving the system of over-determined PDEs derived in Step 4 by use of Mathematica, we would end up with the explicit expressions for a_0, a_i, b_i, c_i, d_i and ξ.

Step 6. Using the results obtained in the above steps, we can derive a series of fundamental solutions of Eq. (3) depending on the solutions $\phi(\xi)$ of Eq. (5). Given different values of h_ρ $(\rho = 0, 1, 2, \ldots, r)$, Eq. (5) has many kinds of solutions, however, the general solutions are difficult to be listed because of the complexity Eq. (5) possesses. In this Letter, we consider the case of $r = 4$. Some special solutions [25–29,36] for this case are listed as follows:

Case 2.1. If $h_3 = h_4 = 0$, Eq. (5) has the following solutions:

$$\phi(\xi) = \sqrt{h_0}\xi, \qquad h_1 = h_2 = 0, \qquad h_0 > 0; \tag{7}$$

$$\phi(\xi) = -\frac{h_0}{h_1} + \frac{1}{4}h_1\xi^2, \qquad h_2 = 0, \qquad h_1 \neq 0; \tag{8}$$

$$\phi(\xi) = -\frac{h_1}{2h_2} + \exp(\sqrt{h_2}\xi), \qquad h_0 = \frac{h_1^2}{4h_2}, \qquad h_2 > 0; \tag{9}$$

$$\phi(\xi) = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2}\sin(\sqrt{-h_2}\xi), \qquad h_0 = 0, \qquad h_2 < 0; \tag{10}$$

$$\phi(\xi) = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2}\sinh(\sqrt{h_2}\xi), \qquad h_0 = 0, \qquad h_2 > 0; \tag{11}$$

Case 2.2. If $h_1 = h_3 = 0$, Eq. (5) has the following solutions:

$$\phi(\xi) = \sqrt{-\frac{h_2}{h_4}}\,\mathrm{sech}(\sqrt{h_2}\xi), \qquad h_0 = 0, \qquad h_2 > 0, \qquad h_4 < 0; \tag{12}$$

$$\phi(\xi) = \sqrt{-\frac{h_2}{2h_4}}\tanh\left(\sqrt{-\frac{h_2}{2}}\xi\right), \qquad h_0 = \frac{h_2^2}{4h_4}, \qquad h_2 < 0, \qquad h_4 > 0; \tag{13}$$

$$\phi(\xi) = \sqrt{-\frac{h_2}{h_4}}\sec(\sqrt{-h_2}\xi), \qquad h_0 = 0, \qquad h_2 < 0, \qquad h_4 > 0; \tag{14}$$