Physics Letters A 378 (2014) 3041–3045

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

On the final configuration of a plane magnetic field dragged by a highly conducting fluid and anchored at the boundary

Manuel Núñez

Sección de Matemáticas and IMUVA, Universidad de Valladolid, 47011 Valladolid, Spain

ARTICLE INFO

Article history:
Received 8 May 2014
Received in revised form 25 July 2014
Accepted 12 August 2014
Available online 28 August 2014
Communicated by A.P. Fordy

Keywords:
Magnetic diffusion
Asymptotic expansion
Boundary layer
Stagnation point

ABSTRACT

The final configuration of the magnetic field dragged by a plane conducting flow such that the feet of the field lines are fixed at the boundary is studied by asymptotic analysis on the small magnetic diffusivity. The first order approximation yields that the streamlines become also magnetic field lines and the magnetic potential satisfies an ordinary differential equation on the transversal variable whose boundary values are found by the addition of a boundary layer. It turns out that these values correspond to certain averages along the boundaries, except when there exist stagnation points, which dominate the magnetic potential diffusion. Corners of the boundary curves behave differently, because stagnation points there disappear after straightening the curve by a change of variables that also kills the zero of the velocity.

1. Introduction

Following the seminal work of Hartmann and Lazarus in [1] about a transversal magnetic field acting as a brake on a conducting viscous flow, the topic of a conducting fluid near a plate has generated an enormous amount, probably even excessive, of literature, accounting for every possibility including suction, heat transfer, gravitational effects, rotation, etc. In the overwhelming majority of these papers the magnetic field is assumed unaffected by the flow, a hypothesis lacking real substance and mainly motivated by the desire to avoid the mathematical complication of adding Maxwell's equations to the problem. Refs. [2–4] are sophisticated contributions to this literature, while e.g. [5–7] also allow for a variable magnetic field. We intend to study the reciprocal problem: taking the flow as independent of the magnetic field, where does the evolution of this lead to? The technique of taking the fluid velocity for granted and considering only the magnetic field evolution as predicted by the induction equation is classical and much used e.g. in kinematic dynamo theory (see e.g. [8]). It is based on the assumption that the Lorentz force may either be ignored because the magnetic field is much weaker than the strength of the flow or because the current density is parallel to the field, as it occurs in force-free configurations. Whatever the reason, the magnetic field $\mathbf{B}$ is governed by the magnetohydrodynamic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \epsilon \Delta \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{1}$$

where $\mathbf{v}$ represents the velocity and ϵ the resistivity (or, in adimensional variables, the magnetic diffusivity). Additionally, $\nabla \cdot \mathbf{B} = 0$, so that $\mathbf{B}$ derives from a vector potential. In analogy to the Hartmann setting, we will consider the two-dimensional case where both $\mathbf{v}$ and $\mathbf{B}$ lie in the XY plane and depend only on (x, y). Then we may take a scalar potential $A(t, x, y)$ such that the vector potential may be written as

$$\mathbf{A} = (0, 0, A). \tag{2}$$

In general (1) is identical to

$$\nabla \times \left(\frac{\partial \mathbf{A}}{\partial t}\right) = \epsilon \nabla \times \Delta \mathbf{A} + \nabla \times \big(\mathbf{v} \times (\nabla \times \mathbf{A})\big), \tag{3}$$

which may be uncurled in simply connected sets to yield

$$\frac{\partial \mathbf{A}}{\partial t} = \epsilon \Delta \mathbf{A} + \big(\mathbf{v} \times (\nabla \times \mathbf{A})\big) + \nabla \Phi, \tag{4}$$

where Φ is a gauge potential. With our hypothesis (2), $\nabla \Phi$ is vertical and independent of z, so that $\nabla \Phi = (0, 0, \lambda(t))$. If we define

$$\Lambda(t) = \int_0^t \lambda(s)\, ds, \tag{5}$$

and take instead of A the new scalar potential $A - \Lambda(t)$, we may omit Φ from the following calculations. Denoting this new potential again by A, we have $\mathbf{v} \times (\nabla \times \mathbf{A}) = (-\mathbf{v} \cdot \nabla A)\hat{z}$, so that (4) becomes a simple transport plus diffusion equation:

E-mail address: mnjmhd@am.uva.es.

http://dx.doi.org/10.1016/j.physleta.2014.08.031
0375-9601/

$$\frac{\partial A}{\partial t} = \epsilon \Delta A - \mathbf{v} \cdot \nabla A. \tag{6}$$

To this equation, initial and boundary conditions must be added. Assuming the domain Ω, the velocity $\mathbf{v}$ and the boundary condition $A|_{\partial\Omega} = f$ to be all independent of time, it is a well known result that the solution of (6) tends to the unique solution of

$$\epsilon \Delta A - \mathbf{v} \cdot \nabla A = 0$$
$$A|_{\partial\Omega} = f \tag{7}$$

(see e.g. [9], pp. 157–164). It is true that the convergence gets slower when ϵ decreases, and we are interested in plasmas of small diffusivity, the most usual in astrophysical and fusion phenomena. Nonetheless, the limit state is given by the solution of the elliptic problem (7). The domain represents the channel where the flow occurs, so that we take it as bounded by two curves Γ_1, Γ_2 immovable by the flow; we also assume that there are no stagnation points ($\mathbf{v} = \mathbf{0}$) inside Ω, although we will admit isolated stagnation points at $\partial\Omega$. Since we do not include regions of inflow and outflow of the fluid, this is assumed to flow cyclically. In fact sources and sinks could be included, since the boundary layers for those are well known (see e.g. [10]), but they would add only unrelated complications to our discussion, whose main interest lies in boundary curves which are themselves streamlines. Since the magnetic field lines are the level sets $A =$ const., the boundary condition $A|_{\partial\Omega} = f$ means that the feet of them are fixed, i.e. that the field is anchored at the boundary. One of the curves Γ_i may disappear if the flow is limited to a simply connected set; in that case we allow for a single stagnation point inside Ω, which is a center of the flow. We will study (7) by means of an asymptotic analysis on the small parameter ϵ. The mathematical treatment is considerably complex, and it is related to the exit problem in stochastic fluid flows. The first and substantial study is [11], where the geometry was Cartesian, with periodicity in one variable, and no critical points of the velocity. Critical points at the boundary with this same geometry were allowed and analyzed in [12], where the asymptotic form of the boundary condition may be found. The problem for flows surging from within the domain and flowing through its boundary is studied in [13]; flows entering and leaving the domain are considered in [14], and the flow-invariant simply connected domain case without critical points (except for the center) is studied in [15]. Most of the necessary mathematics (except for the simple connection hypothesis) for our study may be found in [16], but we will simplify and reconstruct some not so clear arguments about the spectral analysis of non-self-adjoint differential operators. Further generalizations occur in [17]. The paper will be mostly self-contained as far as the equations satisfied by the leading order term, but for the asymptotic expression of the boundary conditions we will refer to [12].

2. The first order approximation

With our hypotheses, streamlines fill Ω. Let them be indexed by the coordinate r, so that the boundaries are given by $\Gamma_i : r = r_i$. The other coordinate s is the arc length along each streamline, going from 0 to the length L_r of the line indexed by r. The origin of s is arbitrary, but taken so that the bijection

$$\bar{\Omega} \to \{(r,s) : r_1 \le r \le r_2,\ 0 \le s \le L_r\}$$
$$(x, y) \to (r, s), \tag{8}$$

is smooth. Although conscious of the notation abuse it involves, we will also denote by $A(r, s)$ the function $A(x(r, s), y(r, s))$. Since A is a solution of (7), in the new coordinates A satisfies

$$\epsilon\left(a_{11}\frac{\partial^2 A}{\partial r^2} + 2a_{12}\frac{\partial^2 A}{\partial r \partial s} + a_{22}\frac{\partial^2 A}{\partial s^2} + c_1\frac{\partial A}{\partial r} + c_2\frac{\partial A}{\partial s}\right) + b\frac{\partial A}{\partial s} = 0$$
$$A(r_1, s) = f_1(s), \qquad A(r_2, s) = f_2(s), \tag{9}$$

where

$$a_{11} = |\nabla r|^2, \qquad a_{12} = \nabla r \cdot \nabla s, \qquad a_{22} = |\nabla s|^2$$
$$c_1 = \Delta r, \qquad c_2 = \Delta s, \qquad b = |\mathbf{v}|. \tag{10}$$

As usual in asymptotic analysis, we start assuming that the solution may be expanded in powers of ϵ:

$$A \sim A_0 + \epsilon A_1 + \epsilon^2 A_2 + \ldots \tag{11}$$

It is proved that this expansion is valid in the previously cited literature. In particular,

$$A = A_0 + O(\epsilon). \tag{12}$$

The term in $O(\epsilon)$ is in general not uniform in Ω, but it may be taken fixed in every compact subset of it. In particular this implies that A_0 tends uniformly to A in the compact subsets of Ω as $\epsilon \to 0$.

Inserting (11) in (10) and equating powers of ϵ, we find for the exponent zero that A_0 satisfies

$$b\frac{\partial A_0}{\partial s} = 0, \tag{13}$$

which means, taking into account that $\mathbf{v} \ne \mathbf{0}$ within the domain, that A_0 is constant in the streamlines, i.e. $A_0 = A_0(r)$. For the exponent one, we get

$$\begin{aligned} b\frac{\partial A_1}{\partial s} &= -\left(a_{11}\frac{\partial^2 A_0}{\partial r^2} + 2a_{12}\frac{\partial^2 A_0}{\partial r \partial s} + a_{22}\frac{\partial^2 A_0}{\partial s^2} \right. \\ &\qquad \left. + c_1\frac{\partial A_0}{\partial r} + c_2\frac{\partial A_0}{\partial s}\right) \\ &= -\left(a_{11}\frac{\partial^2 A_0}{\partial r^2} + c_1\frac{\partial A_0}{\partial r}\right). \end{aligned} \tag{14}$$

This may be seen as a definition of A_1, but for it to be correct we must take into account that A_1 must be L_r-periodic in the variable s. Hence the integral of $\partial A_1/\partial s$ between 0 and L_r must be zero, which following (14) means

$$\left(\int_0^{L_r} \frac{a_{11}}{b}\, ds\right)\frac{\partial^2 A_0}{\partial r^2} + \left(\int_0^{L_r} \frac{c_1}{b}\, ds\right)\frac{\partial A_0}{\partial r} = 0. \tag{15}$$

Since b may be small at some points, it is convenient to normalize (15) by dividing both terms by the integral of the inverse of the velocity size. Thus, defining the averages

$$\alpha(r) = \left(\int_0^{L_r} \frac{a_{11}}{b}\, ds\right)\left(\int_0^{L_r} \frac{1}{b}\, ds\right)^{-1}$$
$$\beta(r) = \left(\int_0^{L_r} \frac{c_1}{b}\, ds\right)\left(\int_0^{L_r} \frac{1}{b}\, ds\right)^{-1}, \tag{16}$$

we reach the final equation satisfied by A_0:

$$\alpha A_0'' + \beta A_0' = 0, \tag{17}$$

where $'$ denotes derivation with respect to the unique variable r of A_0. Also

$$|\alpha| \le \sup|a_{11}|, \qquad |\beta| \le \sup|c_1|. \tag{18}$$