Physics Letters A 374 (2010) 3599–3602

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Optimal choices of reference for a quasi-local energy

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ARTICLE INFO

Article history:
Received 4 June 2010
Received in revised form 2 July 2010
Accepted 2 July 2010
Available online 14 July 2010
Communicated by P.R. Holland

Keywords:
Quasi-local energy
Covariant Hamiltonian
Hamiltonian boundary term
Reference spacetime

ABSTRACT

We propose a program for determining the reference for the covariant Hamiltonian boundary term quasi-local energy and test it on spherically symmetric spacetimes. For different observers the quasi-local energy can be positive, zero, or even negative, however the maximum value is positive for both the Schwarzschild and FLRW spacetimes.

1. Introduction

An outstanding fundamental problem in general relativity is that there is no proper definition for the energy density of gravitating systems. (This can be understood as a consequence of the equivalence principle.) The modern concept is that gravitational energy should be non-local, more precisely *quasi-local*, i.e., it should be associated with a closed two-surface (for a comprehensive review see [1]). Here we consider one proposal based on the covariant Hamiltonian formalism [2] wherein the quasi-local energy is determined by the Hamiltonian boundary term. For a specific spacetime displacement vector field on the boundary of a region (which can be associated with the observer), the quasi-local energy—defined as the value of the Hamiltonian boundary term—depends not only on the dynamical values of the fields on the boundary but also on the choice of reference values for these fields. Thus a principal issue in this formalism is the proper choice of reference spacetime for a given observer. Here we test a specific proposal for fixing the Hamiltonian boundary term reference values.

It is generally accepted that the total energy for an asymptotically flat gravitating system should be non-negative and should vanish only for Minkowski space (this is required for stability, see, e.g., [3]; for proofs of this property for GR see [4,5]). In view of this it has been natural to regard these properties as also being desirable for a good quasi-local energy [1,6,7]. The idea that a suitable reference should be the one which gives the minimal energy then followed quite naturally. We have proposed using this approach to choose the optimal reference for the covariant Hamiltonian boundary term. Here we consider this optimal choice program for both static and dynamic spherically symmetric spacetimes (the most important test cases). Specifically, for the Schwarzschild and the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes, we found the resultant quasi-local energies can be positive, zero, *or even negative* for different observers. However, for both cases, there is one observer who would measure the maximum energy, and for this observer the associated energy is positive. Furthermore we find that this energy-extremization program (at least for these spherically symmetric systems) is equivalent to matching the geometry at the two-sphere boundary, which provides for a simple interpretation of the displacement vector.

2. The Hamiltonian formulation

We begin with a brief review of the covariant Hamiltonian formalism as developed by our research group [8–12] (for some additional developments along similar lines see [13,14]). A *first order Lagrangian 4-form* for a k-form field φ can be expressed as

$$\mathcal{L} = d\varphi \wedge p - \Lambda(\varphi, p). \tag{1}$$

The action should be invariant under local diffeomorphisms, which infinitesimally correspond to a displacement along some vector field N. From Noether's theorem there is a conserved translational

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2010.07.004

current which can be written as a 3-form linear in the displacement vector plus a total differential:

$$\mathcal{H}(N) := \pounds_N\varphi \wedge p - i_N\mathcal{L} := N^\mu \mathcal{H}_\mu + d\mathcal{B}(N). \tag{2}$$

Here $\mathcal{H}_\mu \equiv -i_\mu\varphi \wedge \frac{\delta\mathcal{L}}{\delta\varphi} + \varsigma\frac{\delta\mathcal{L}}{\delta p} \wedge i_\mu p$ with $\varsigma := (-1)^k$; this identity is a necessary consequence of *local* diffeomorphism invariance (i.e., symmetry for non-constant N^μ). Consequently $\mathcal{H}_\mu$ vanishes on shell; hence the value of the Hamiltonian—the conserved quantity associated with a local displacement N and a spatial region Σ—is determined by a 2-surface integral over the region's boundary:

$$E(N,\Sigma) := \int_\Sigma \mathcal{H}(N) = \oint_{\partial\Sigma} \mathcal{B}(N). \tag{3}$$

For *any* choice of N this expression defines a conserved quasi-local quantity. Different choices of boundary term correspond to different boundary conditions.

Einstein's gravity theory, general relativity (GR), can be formulated in several ways. For our purposes the most convenient is to take the *orthonormal coframe* $\vartheta^\mu = \vartheta^\mu{}_k dx^k$ and the *connection one-form* $\omega^\alpha{}_\beta = \Gamma^\alpha{}_{\beta k} dx^k$ as the geometric potentials. Moreover we take the connection to be *a priori* metric compatible: $Dg_{\alpha\beta} := dg_{\alpha\beta} - \omega^\gamma{}_\alpha g_{\gamma\beta} - \omega^\gamma{}_\beta g_{\alpha\gamma} \equiv 0$. Restricted to orthonormal frames where the metric components are constant, this condition reduces to the algebraic constraint $\omega^{\alpha\beta} \equiv \omega^{[\alpha\beta]}$.

We consider the vacuum (source free) case for simplicity. GR can be obtained from the first order Lagrangian 4-form

$$\mathcal{L}_{\mathrm{GR}} = \Omega^{\alpha\beta} \wedge \rho_{\alpha\beta} + D\vartheta^\mu \wedge \tau_\mu - V^{\alpha\beta} \wedge \left(\rho_{\alpha\beta} - \frac{1}{2\kappa}\eta_{\alpha\beta}\right), \tag{4}$$

where $\Omega^\alpha{}_\beta := d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$ is the *curvature* 2-form, $D\vartheta^\mu := d\vartheta^\mu + \omega^\mu{}_\nu \wedge \vartheta^\nu$ is the *torsion* 2-form, and $\eta^{\alpha\beta\cdots} := {}^\star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \cdots)$ is the dual form basis. The 2-forms $\Omega^{\alpha\beta}$, $V^{\alpha\beta}$ and $\rho_{\alpha\beta}$ are antisymmetric. We take $\kappa := 8\pi G/c^4 = 8\pi$. Several possible boundary terms were identified, each associated with a distinct type of boundary condition. In [12] a "preferred boundary term" (it has a certain covariant property, directly gives the Bondi energy flux, and has a positive total energy proof) for GR was identified:

$$\mathcal{B}(N) = \frac{1}{16\pi}\left(\Delta\omega^\alpha{}_\beta \wedge i_N\eta_\alpha{}^\beta + \bar{D}_\beta N^\alpha \Delta\eta_\alpha{}^\beta\right), \tag{5}$$

where Δ indicates the difference between the dynamic and reference values and $\bar{D}_\beta$ is the reference covariant derivative. The reference values can be determined by pullback from an embedding of the boundary into a suitable reference space.

3. The energy-extremization program

Here we explicitly formulate the extremization program for static spherically symmetric spacetimes. The Schwarzschild-like metric in "standard" spherical coordinates is given by

$$ds^2 = -A\,dt^2 + A^{-1}\,dr^2 + r^2\,d\Omega_2^2, \tag{6}$$

where $A = A(r)$ and $d\Omega_2^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$. However, there are other favorable coordinate choices for the Schwarzschild metric (e.g., Painlevé–Gullstrand, Eddington–Finkelstein, Kruskal–Szekeres). In order to accommodate most well-known coordinates, we consider a more general version of the Schwarzschild-like metric via a coordinate transformation $t = t(u,v)$, $r = r(u,v)$; the metric becomes

$$\begin{aligned} ds^2 = {} & -\left(At_u^2 - A^{-1}r_u^2\right)du^2 + 2\left(A^{-1}r_u r_v - At_u t_v\right)du\,dv \\ & + \left(A^{-1}r_v^2 - At_v^2\right)dv^2 + r^2\,d\Omega_2^2. \end{aligned} \tag{7}$$

The Minkowski spacetime

$$d\bar{s}^2 = -dT^2 + dR^2 + R^2\,d\Theta^2 + R^2\sin^2\Theta\,d\Phi^2 \tag{8}$$

is a natural choice for the reference. However, the essential issue of the reference choice is the identification between the reference and physical spacetime coordinates. A legitimate approach for the spherically symmetric case is to assume $T = T(u,v)$, $R = R(u,v)$, $\Theta = \theta$, $\Phi = \varphi$ along with $R_0 := R(t_0, r_0) = r_0$; this symmetrically embeds a neighborhood of the two-sphere boundary S at (t_0, r_0) into the Minkowski reference such that the two-sphere boundary is embedded isometrically. Assume that the displacement vector

$$N = N^u\partial_u + N^v\partial_v = N^t\partial_t + N^r\partial_r = N^T\partial_T + N^R\partial_R \tag{9}$$

is future timelike and the orientation is preserved under diffeomorphisms, i.e., $\sqrt{-\alpha} := t_u r_v - t_v r_u > 0$ and $X^{-1} := T_u R_v - T_v R_u > 0$. The second term of Eq. (5) vanishes for spherically symmetric spacetimes; the quasi-local energy can then be evaluated to be

$$E = \frac{r}{2}\left(N^u B + N^v C\right)\sqrt{-\alpha}, \tag{10}$$

$$B = XT_u + g^{vu}(R_u - 2r_u) + g^{vv}(R_v - 2r_v), \tag{11}$$

$$C = XT_v + g^{uu}(2r_u - R_u) + g^{uv}(2r_v - R_v), \tag{12}$$

where the subscripts indicate partial differentiations. Note that the quasi-local energy is evaluated on the boundary two-sphere S; the variables appearing in Eq. (10) and in the following are also evaluated on S. Each choice of the embedding variables $\{T_u, T_v, R_u, R_v\}$ means a different embedding, hence a different reference. For any given displacement vector we extremize the energy with respect to the embedding variables; we get four equations, but only three are independent:

$$N^u R_u + N^v R_v = N^R = 0, \tag{13}$$

$$X^2 T_v\left(N^u T_u + N^v T_v\right) - \alpha^{-1}\left(g_{uv}N^u + g_{vv}N^v\right) = 0, \tag{14}$$

$$X^2 T_u\left(N^u T_u + N^v T_v\right) - \alpha^{-1}\left(g_{uu}N^u + g_{uv}N^v\right) = 0. \tag{15}$$

A useful combination (15) $\times R_v -$ (14) $\times R_u$ gives

$$\begin{aligned} & X\left(N^u T_u + N^v T_v\right) + \alpha^{-1}\left[\left(g_{uv}N^u + g_{vv}N^v\right)R_u \right. \\ & \quad \left. - \left(g_{uu}N^u + g_{uv}N^v\right)R_v\right] = 0. \end{aligned} \tag{16}$$

From Eq. (13) we get

$$R_u = -\frac{N^v}{N^u}R_v, \qquad N^T := N^u T_u + N^v T_v = \frac{N^u}{XR_v}; \tag{17}$$

then R_v can be found from Eq. (16):

$$\frac{N^u}{R_v} - \alpha^{-1}\frac{R_v}{N^u}g(N,N) = 0 \quad \Rightarrow \quad R_v^2 = \frac{\alpha(N^u)^2}{g(N,N)}. \tag{18}$$

We require the displacement vector to be future timelike, i.e., $N^T > 0$ and $N^u > 0$, and the orientation to be preserved, i.e., the Jacobians are positive. Then R_v should be positive, and therefore

$$R_v = \sqrt{\frac{\alpha}{g(N,N)}}N^u, \qquad R_u = -\sqrt{\frac{\alpha}{g(N,N)}}N^v. \tag{19}$$

Now we calculate the energy. Using Eq. (19) we get

$$\sqrt{-\alpha}\left(N^u B + N^v C\right) = 2\left(\sqrt{-g(N,N)} - AN^t\right), \tag{20}$$

where the explicit metric is used in the calculation. Choose N to be unit timelike on the two-sphere, i.e., $-1 = g(N,N) = g_{uu}(N^u)^2 + 2g_{uv}N^u N^v + g_{vv}(N^v)^2$, then the quasi-local energy for any given future timelike displacement vector N reduces to

$$E = r\left(1 - AN^t\right), \tag{21}$$