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Obtaining non-Abelian field theories via the Faddeev–Jackiw symplectic formalism

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ABSTRACT

In this Letter we construct non-Abelian field theories employing the Faddeev–Jackiw symplectic formalism. The original Abelian fields were modified in order to introduce the non-Abelian algebra. We construct the $SU(2)$ and $SU(2)\otimes U(1)$ Yang–Mills theories having as starting point the $U(1)$ Maxwell electromagnetic theory.

1. Introduction

The main objective of this Letter is not to present new results but to present a new path for the well-known Faddeev–Jackiw (FJ) method which is understood so far as an approach that leads with constrained systems. The relevance of this new insight resides in the property, demonstrated precisely here, of introducing non-Abelianity into Abelian models. We show this property using only the $U(1)$ Maxwell theory as the starting model and, using the standard FJ formalism, we obtained two non-Abelian models, the $SU(2)$ and $SU(2)\otimes U(1)$ Yang–Mills theories. Since non-Abelian theories are important in high energy physics, we believe that this new path can change the way of fathoming the FJ formalism.

The consistent quantization method for constrained systems was introduced by Dirac in 60s [1]. In particular, the Dirac formalism analyzes the canonical structures of any theory, which are essential to the development of quantum theories. In [2] Faddeev and Jackiw suggested a symplectic approach for constrained systems based on a first-order Lagrangian.

After this work, Barcelos-Neto and Wotzasek (BW) [3] extended the FJ symplectic formalism to the case where the constraints cannot be completely eliminated and several models were used to exemplify the method. The method has the key point that these constraints produce deformations in the two-form symplectic matrix in such way that, when all constraints are considered, the symplectic matrix is non-singular. As a result, the authors have directly obtained the Dirac brackets. It is important to mention that sometimes, it happens that the two-form matrix is singular and no new constraint can be obtained from the corresponding zero-mode. This is the case when one deals with gauge theories. At this point one introduces convenient gauge conditions like a constraint and the two-form matrix becomes, therefore, invertible.

In recent works, the FJ symplectic formalism has been used in a systematic way with different purposes: to study hidden symmetries; in the construction of gauge theories (duality); noncommutative gauge theories; to solve the obstruction problem for the introduction of the canonical Lagrangian formulation for rotational systems [4]; to analyze the quantization of lightcone QCD [5]; to study the constraints appearing in a noncommutative version of the Hall effect [6]; to consider FRW metric model in quantum cosmology [7] and many others that can be found in Faddeev–Jackiw's literature.

The first published work generalizing the $U(1)$ Maxwell electromagnetic theory that requires a non-Abelian group structure was introduced by Klein in [8]. It took 15 years until Yang and Mills were able to provide the full account of a $SU(2)$ non-Abelian gauge theory [9].

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The purpose of the present work is to show that the FJ symplectic formalism may be used in order to obtain, in a systematic way, non-Abelian field theories. The formalism presented here has the following methodology: firstly, the original Abelian fields are changed in order to introduce the non-Abelian algebra; secondly, the FJ symplectic method is implemented and the gauge symmetry that belongs to some non-Abelian symmetry group, $SU(2)$ for instance, is introduced. This is carried out through the introduction of a convenient symplectic zero-mode matrix, which must be considered singular. Thus, it is possible to build the new one-form tensor and hence, the canonical momenta are obtained. We note that new terms arise due to the new symmetry group considered.

2. From $U(1)$ Maxwell Electromagnetic Theory to $SU(2)$ Yang–Mills theory

In order to introduce this methodology, let us consider the $U(1)$ Maxwell Electromagnetic Theory in four dimensions, whose dynamics is governed by the following Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the space–time metric we are considering is $g_{00} = +1$ and $g_{ii} = -1$, with $i = 1, 2, 3$.

To input an internal symmetry group, the original field changes as

$$A_\mu \to A^a_\mu, \tag{2}$$

where "a" denote an index belonging to some internal symmetry group introduced conveniently into the original theory. Thus, let us rewrite the original Lagrangian density as

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}, \tag{3}$$

where $G^a_{\mu\nu}$ is an arbitrary tensor that will be obtained later. Notice that we have to transform

$$G^a_{\mu\nu} \to F^a_{\mu\nu}, \tag{4}$$

if we want to reduce the new symmetry group to the original group. Considering this result, let us look for a $G^a_{\mu\nu}$ term with the following form

$$G^a_{\mu\nu} = F^a_{\mu\nu} + g\tilde{F}^a_{\mu\nu}, \tag{5}$$

where g is a parameter and $\tilde{F}^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}(A^a_\alpha)$ is an arbitrary antisymmetric tensor. We will also assume that the fields A^a_μ satisfy a usual non-Abelian algebra.

Now, in order to carry out the second step of the method, we will first rewrite the Lagrangian density in (3) in its first-order form using the canonical momenta

$$\pi^b_0 = 0, \tag{6}$$

$$\pi^b_j = -\partial_0 A^b_j + \partial_j A^b_0 - g\tilde{F}^b_{0j}. \tag{7}$$

Hence, the first-order Lagrangian density can be written as

$$\mathcal{L} = \pi^a_i \dot{A}^a_i - \frac{1}{2}\pi^a_i \pi^a_i - \partial_i \pi^a_i A^a_0 + g\tilde{F}^a_{0i}\pi^a_i - \frac{1}{4}\left(F^a_{ij}F^{aij} + 2gF^{aij}\tilde{F}^{aij} + g^2\tilde{F}^a_{ij}\tilde{F}^{aij}\right). \tag{8}$$

The symplectic variables are

$$\xi^\alpha = \left(A^a_i, \pi^a_i, A^a_0\right), \tag{9}$$

and the two-form matrix is given by

$$f = \begin{pmatrix} \frac{\delta\pi^a_i}{\delta A^b_j} - \frac{\delta\pi^b_j}{\delta A^a_i} & -\delta_{ij}\delta^{ab}\delta^{(4)}(x-y) & 0 \\ \delta_{ji}\delta^{ba}\delta^{(4)}(x-y) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

It is obvious that this matrix is singular and can have the following zero-mode $\nu = (0\ 0\ 1)$. This zero-mode leads to the Gauss law constraint

$$\Omega^a \equiv D^{abi}\pi^b_i, \tag{11}$$

where D^{abi} is the operator given by

$$D^{abi} \equiv \delta^{ab}\partial^i + \frac{\delta\tilde{F}^{b0i}}{\delta A^a_0}. \tag{12}$$

Following the prescription of the symplectic formalism, the constraint Ω^a will be incorporated into the Lagrangian density in order to make the new Lagrangian density as