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Lagrangian formulation of noncommutative fluid models

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ABSTRACT

In this work we study Lagrangian formulations for the noncommutative versions of the irrotational and rotational fluid models. These formulations will be obtained by using the Faddeev–Jackiw symplectic formalism. In this context, interesting results will be revealed, for example, a chiral behavior into both fluid models comes up. In fact, distinct and non-dynamically equivalent Lagrangian descriptions of the noncommutative versions of the fluid models can be proposed.

1. Introduction

In this Letter, we will use the symplectic approach to reveal the noncommutativity that lives within some models [1]. This technique will be applied to the irrotational and rotational nonrelativistic fluid models [2].

Some years ago, Bordemann and Hoppe [3] demonstrated that the relativistic theory of membranes are given by integrable systems by reducing the problem to a 2-dimensional fluid dynamics. In that paper the authors simplify the light-cone gauge description of a relativistic membrane [4] moving into a Minkowski space by transforming independent into dependent variables. This fact allowed them to find explicit solution to all constraints and to reduce the original system of field equations, with four functions, into a system with only two functions. The dynamics is governed by a Hamiltonian reduced to a $SO(1,3)$ invariant $(2+1)$-dimensional theory of isentropic fluid dynamics, where the pressure is inversely proportional to the mass-density. Unfortunately, this procedure does not preserve the gauge symmetry exhibited initially by the relativistic theory of membranes, which can cast doubt on the reduction procedure.

Afterward, the study of the nonrelativistic fluid mechanics model [2] has attracted too much attention [5–10]. This subject is of broader interest since it also offers connections with the hydrodynamical description of quantum mechanics [11,12], parton model [5], black-hole cosmology [13] and hydrodynamics of superfluid systems [14]. Most of these investigations are focused on finding the solutions of this Galileo invariant system in d-dimensions in connection with the solutions of the relativistic d-brane system in $(d+1)$-dimensions [6,7], which is of direct interest to theoretical particle physics. In particular, this last point was responsible for the recent spate of interest in clarifying the presence of a hidden dynamical Poincaré symmetry of this non-relativistic model realized by field dependent diffeomorphism. In terms of the canonical variables one can compute the Poisson algebra and to reproduce the Poincaré algebra for a system (membrane) in one dimension higher [7]. In this scenario, Neves et al. [15] had shown that the fluid model has a dynamically equivalent set of gauge symmetries and lift the global extra symmetries [5–7] into local ones.

Noncommutative field theories provide fruitful avenues of exploration for several reasons [16,17]. Firstly, some quantum field theories have a better behavior in noncommutative spacetime than in ordinary spacetime. In fact, some are completely finite, even non-perturbatively. Thus, spacetime noncommutativity presents itself as an alternative to string theory or supersymmetry. Secondly, it is a useful arena for studying physics beyond the standard model and also for standard physics in strong external fields. Thirdly, it sheds light on alternative underlying issues in quantum field theory. For instance, renormalization and axiomatic programmes. Finally, it naturally relates field theory to gravity. Since the field theory may be quantized, this may provide significant insights into the problem of quantizing gravity.

It is known that d-brane in $(d+1)$-dimensions has a noncommutative nature when interaction is considered due to the presence of external electromagnetic field [18–20]. In this context, the presence of a hidden dynamical Poincaré symmetry of the non-relativistic fluid

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model *via* field dependent diffeomorphism does not work. In the present Letter, we will show that this noncommutativity also manifest itself in fluid model.

In order to become the work self-contained, a brief review of noncommutative (NC) symplectic formalism will be reviewed in the next section. In Sections 3 and 4, respectively, noncommutative representations of irrotational and rotational fluid models will be presented. In the last section, our conclusions and further discussions will be pointed out.

2. The NC generalized symplectic formalism

The quantization by deformation [21] consists of the substitution of the canonical quantization process by the algebra $\mathcal{A}_\hbar$ of quantum observables generated by the same classical one obeying the Moyal product. Namely, the canonical quantization

$$\{h,g\}_{PB}=\frac{\partial h}{\partial\zeta_a}\omega_{ab}\frac{\partial g}{\partial\zeta_b}\longrightarrow\frac{1}{\imath\hbar}[\mathcal{O}_h,\mathcal{O}_g],\tag{1}$$

with $\zeta=(q_i,p_i)$, is replaced by the $\hbar$-star deformation of $\mathcal{A}_0$, given by

$$\{h,g\}_\hbar=h*_\hbar g-g*_\hbar h,\tag{2}$$

where

$$(h*_\hbar g)(\zeta)=\exp\left\{\frac{\imath}{2}\hbar\omega_{ab}\partial^a_{(\zeta_1)}\partial^b_{(\zeta_2)}\right\}h(\zeta_1)g(\zeta_2)|_{\zeta_1=\zeta_2=\zeta},\tag{3}$$

with $a,b=1,2,\dots,2N$ and with the following classical symplectic structure

$$\omega_{ab}=\begin{pmatrix}0&\delta_{ij}\\-\delta_{ji}&0\end{pmatrix}\quad\text{with } i,j=1,2,\dots,N,\tag{4}$$

that satisfies the relation below

$$\omega^{ab}\omega_{bc}=\delta^a_c.\tag{5}$$

The quantization by deformation can be generalized assuming a generic classical symplectic structure Σ^{ab}. In this way the internal law will be characterized by $\hbar$ and by another deformation parameter (or more). As a consequence, the Σ-star deformation of the algebra becomes

$$(h*_{\hbar\Sigma} g)(\zeta)=\exp\left\{\frac{\imath}{2}\hbar\Sigma_{ab}\partial^a_{(\zeta_1)}\partial^b_{(\zeta_2)}\right\}h(\zeta_1)g(\zeta_2)|_{\zeta_1=\zeta_2=\zeta},\tag{6}$$

with $a,b=1,2,\dots,2N$.

This new star-product generalizes the algebra of the symplectic variables in the following way

$$\{h,g\}_{\hbar\Sigma}=\imath\hbar\Sigma_{ab}.\tag{7}$$

In [22,23] the authors proposed a quantization process in order to construct a bridge between the NC classical mechanics and the NC quantum mechanics, through the generalized Dirac quantization,

$$\{h,g\}_\Sigma=\frac{\partial h}{\partial\zeta_a}\Sigma_{ab}\frac{\partial g}{\partial\zeta_b}\longrightarrow\frac{1}{\imath\hbar}[\mathcal{O}_h,\mathcal{O}_g]_\Sigma.\tag{8}$$

The relation above can also be obtained through a particular transformation of the usual classical phase space, namely,

$$\zeta'_a=T_{ab}\zeta^b,\tag{9}$$

where the transformation matrix is

$$T=\begin{pmatrix}\delta_{ij}&-\frac{1}{2}\theta_{ij}\\\frac{1}{2}\beta_{ij}&\delta_{ij}\end{pmatrix},\tag{10}$$

where θ_{ij} and β_{ij} are antisymmetric matrices. As a consequence, the original Hamiltonian transforms

$$\mathcal{H}(\zeta_a)\longrightarrow\mathcal{H}(\zeta'_a),\tag{11}$$

where the corresponding symplectic structure is

$$\Sigma_{ab}=\begin{pmatrix}\theta_{ij}&\delta_{ij}+\sigma_{ij}\\-\delta_{ij}-\sigma_{ij}&\beta_{ij}\end{pmatrix},\tag{12}$$

with $\sigma_{ij}=-\frac{1}{8}[\theta_{ik}\beta_{kj}+\beta_{ik}\theta_{kj}]$. Due to this, the commutator relations can be written as

$$[q'_i,q'_j]=\imath\hbar\theta_{ij},\qquad[q'_i,p'_j]=\imath\hbar(\delta_{ij}+\sigma_{ij}),\qquad[p'_i,p'_j]=\imath\hbar\beta_{ij}.\tag{13}$$

It is clear that in this Letter we are only analyzing systems where the symplectic algebra of Eq. (13) involves only constants. It is worthwhile to mention that there are systems where the symplectic algebra (Eq. (13)), involves phase space dependent quantities, rather than just constants. For instance we can mention the noncommutative Landau problem (for example in [24] and references therein). A particle in the noncommutative plane, coupled to a constant magnetic field and an electric potential will possess an algebra similar to