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A competitive game whose maximal Nash-equilibrium payoff requires quantum resources for its achievement

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ABSTRACT

While it is known that shared quantum entanglement can offer improved solutions to a number of purely cooperative tasks for groups of remote agents, controversy remains regarding the legitimacy of quantum games in a competitive setting. We construct a competitive game between four players based on the minority game where the maximal Nash-equilibrium payoff when played with the appropriate quantum resource is greater than that obtainable by classical means, assuming a local hidden variable model.
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1. Introduction

Game theory is a branch of mathematics dealing with strategic interactions of competitive agents where the outcome is contingent upon the combined actions of the agents. In 1999, game theory was formally extended into the quantum realm by replacing the classical information with qubits and the player actions by quantum operators [1,2]. Since then much work has been done in the new discipline of quantum game theory [3,4] and attempts have been made to put it on a more formal footing [5]. There have been objections that quantum games are not truly quantum mechanical and have little to do with the underlying classical games [6–8]. However, attempts have been made to counter these arguments [9]. In addition, quantum games have been shown to be more efficient than classical games, in terms of information transfer, and that finite classical games are a proper subset of quantum games [10], thus demonstrating that not all quantum games can be reduced to classical ones. Quantum game protocols have also been proposed that use the non-local features of quantum mechanics which have no classical analogue [11,12].

In the present work we construct a game that is a minimal quantum generalization of a possible classical game and that has a Nash equilibrium that is not achievable by any classical hidden variable model. Our model is distinguished by its competitive nature from situations, also referred to in the literature as quantum games, that involve a number of agents solving a cooperative task by quantum means [13,14]. Since Van Enk's criticism of quantum games [6] there have been many attempts to distinguish quantum games from classical ones. For example, some authors have shown that the payoff in a quantum game can be separated into a pseudo-classical term and a term dependent on quantum interference [15,16]. The size of the interference term is dependent on the level of entanglement between the strategies of the players, and is therefore an artifact of the quantum nature of the game. However, the presence of quantum interference alone does not prove that the quantum equilibrium could not be obtained by some classical means, e.g., communication or the use of a trusted third party. For example, we show later that the Nash equilibrium for the four-player quantum minority game, though having a superior payoff than that of a standard classical minority game, can be obtained classically using a trusted third party who communicates with the players prior to their moves. In the following section we construct a game with a Nash equilibrium that can only be reached using quantum entanglement, and that cannot be obtained with *any* classical resources, with the exception of communication during the game along with enforceable agreements, which are beyond the scope of competitive game theory. This is analogous to a Bell inequality, whose violation can only be realised through the non-locality inherent in quantum entanglement. This result is important in confirming the legitimacy of quantum games.

The minority game was introduced in 1997 [17] as a simple multi-agent model that is able to reproduce much of the behaviour of financial markets. The agents independently select one of two choices ('buy' or 'sell') and those in the minority win, the idea being that when everyone is buying prices are inflated and it is best

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to be a seller and vice versa. In a one-shot minority game the best players can do is to select among the alternatives at random with an unbiased coin. The simplest non-trivial situation is the four-player game: only one player can win; however, there is a fifty percent chance that there is no minority, in which case all players receive zero payoff. Versions of the minority game utilizing quantum resources have attracted attention since the probability of the no-minority case can be eliminated in the four player game [18], or reduced for even $N > 4$ [19]. These result are robust even in the presence of decoherence [20]. In addition, utilizing a particular set of tunable four-party entangled states as the quantum resource shared by the players, there is an equivalence between the optimal game payoffs and the maximal violation of the four-party MABK-type Bell inequality [21–23] for the initial state [24].

In the quantum minority game each player receives one qubit from a known entangled state. They can act on their qubit with a local unitary operator. The qubits are then measured in the computational basis, and payoffs are awarded as in the classical game. In the four player game, starting with the GHZ state

$$\frac{1}{\sqrt{2}}\big(\hat{I}^{\otimes 4} + \mathrm{i}\hat{\sigma}_x{}^{\otimes 4}\big)|0000\rangle = \frac{1}{\sqrt{2}}\big(|0000\rangle + \mathrm{i}|1111\rangle\big), \tag{1}$$

if each player operates with

$$\hat{s} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\mathrm{i}\pi/16} & \mathrm{i}e^{-\mathrm{i}\pi/16} \\ \mathrm{i}e^{\mathrm{i}\pi/16} & e^{-\mathrm{i}\pi/16} \end{pmatrix}, \tag{2}$$

the resulting superposition contains only those states where one of the four players is in the minority and so the average payoff, $\langle\$\rangle$ is $\frac{1}{4}$, compared with $\frac{1}{8}$ for the classical game [18]. When the strategy[1] of all players is to select this operator with certainty, the result is a Nash equilibrium, a strategy profile from which no player can improve their payoff by a unilateral change in strategy. The result is also Pareto optimal, one from which no player can improve their payoff without someone else being worse off.

One complaint, however, that can be leveled at the quantum versions is that the same outcome can be achieved by a purely classical local hidden variable model. For example, in a four-player minority game a trusted third party could choose one of the eight classical messages 0001, 0010, 0100, 1000, 1110, 1101, 1011, or 0111 at random and then inform each of the players of their selected value. None of the players has an incentive to vary their choice, and the expected payoff is fair to all players. Such an arrangement would also yield $\langle\$\rangle = \frac{1}{4}$.

In this Letter we introduce a competitive game having a Nash-equilibrium maximal payoff that requires the use of quantum resources (i.e., it cannot be achieved with resources whose statistical properties can be modeled using local hidden variables). This is in contrast to previous work on cooperative games such as the XOR game, odd cycle game, and magic square game [13]: even though those games were also shown to be equivalent to a corresponding Tsirelson-type inequality and can be used to demonstrate a Bell inequality, the critical distinction is that the game we consider has a competitive aspect.

2. Definition of the game

We now define the game that is the subject of this Letter. This four-player game will be based partly on the minority game and partly on what we call the *anti-minority game*. While the minority game provides a payoff of 1 for the player who answers differently from the other three (if there is exactly one such player) and no payoff to any other player, the anti-minority game rewards the case where there is no minority, providing a payoff of $\frac{1}{4}$ to all players when all players give the same answer or there is a 50/50 split. That is, all the players score $\frac{1}{4}$ on just those occasions when there would be no winner in a minority game.

The overall game is a combination of these two games. The players do not know beforehand whether the payoff matrix will be that of the minority game or that of the anti-minority game. The players are allowed to meet privately before the game to discuss a joint strategy and, if they wish, prepare physical resources (classical or quantum) for each of them to bring with them to the game. The players are subsequently isolated and prevented from communicating for the rest of the game. An impartial referee (someone other than the players) then asks each of the isolated players one of two questions: either, “What is the value of X?” or, “What is the value of Z?” to which the player must respond with either $+1$ or -1 as she chooses. Each player may, if she wishes, use whatever physical resource she brought with her to aid in answering her question. The game being played (minority or anti-minority)—and thus the payoff matrix—is determined by the set of questions asked by the referee. If the referee has asked three of the players for the value of Z and one of the players for the value of X, then the players are playing the minority game. If the referee asks three of the players for the value of X and one player for the value of Z, then the payoff matrix is that of the anti-minority game. The referee has chosen the question list uniformly at random from the following chart before the game begins:

$$\left.\begin{matrix} X_1Z_2Z_3Z_4 \\ Z_1X_2Z_3Z_4 \\ Z_1Z_2X_3Z_4 \\ Z_1Z_2Z_3X_4 \end{matrix}\right\} \quad \text{minority game;}$$

$$\left.\begin{matrix} Z_1X_2X_3X_4 \\ X_1Z_2X_3X_4 \\ X_1X_2Z_3X_4 \\ X_1X_2X_3Z_4 \end{matrix}\right\} \quad \text{anti-minority game.}$$

Thus, each of these lists has probability $\frac{1}{8}$ of being asked by the referee. Only these question lists are used. Notice that once the payoff matrix is fixed (by the total number of each question asked), there is no further dependence on which player was asked which question, with the payoff determined entirely by the players' answers (of ± 1).

The list $X_1Z_2Z_3Z_4$, for instance, represents player 1 being asked for the value of X and players 2–4 being asked for the value of Z. According to the chart, this corresponds to the minority game. Now let's say, for example, that player 3 answers $+1$, while players 1, 2, and 4 answer -1. Then player 3 receives a payoff of 1, and the others receive nothing. It does not matter which question player 3 was asked, only that his answer ($+1$) is different from the others' (-1) and that the game being played is the minority game.

3. Bounds on expected payoff regardless of strategy

In devising a strategy for this overall game the challenge is that the players don't know *a priori* whether they are playing the (competitive) minority game or the (cooperative) anti-minority game. Since all eight possibilities have equal probabilities, the two games are equally likely.

At this point, it is useful to examine the bounds on a player's expected payoff *regardless of strategy*. Whether such a payoff is achievable by any particular strategy is a separate—and important—question that will be addressed shortly. But there are a few things that can be stated about the game that must hold for any strategy:

[1] A *strategy* in game theory is a complete prescription of a player's actions, including all contingencies.

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