Physics Letters A 374 (2010) 3629–3634

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

On the periodic solutions for both nonlinear differential and difference equations: A unified approach

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ARTICLE INFO

Article history:
Received 28 May 2010
Received in revised form 29 June 2010
Accepted 5 July 2010
Available online 10 July 2010
Communicated by R. Wu

Keywords:
Differential and difference equation
Hirota's bilinear method
Riemann theta function
Periodic wave solution
Soliton solution

ABSTRACT

A direct and unifying scheme for disclosure of periodic wave solutions of both nonlinear differential and difference equations is presented. The scheme is based on Hirota's bilinear form and certain Riemann theta function formulae. The relations between the periodic wave solutions and soliton solutions are rigorously established.

1. Introduction

The bilinear derivative method developed by Hirota is a powerful and direct approach to construct exact solution of nonlinear equations. Once a nonlinear equation is written in bilinear forms by a dependent variable transformation, then multi-soliton solutions are usually obtained [1–5]. Based on the use of Hirota method and theta function identities, one of the authors, Chow proposed a method to obtain doubly periodic solutions in terms of rational function of theta functions [6–9]. Nakamura proposed a convenient way to construct a kind of quasi-periodic solutions of nonlinear equations, where the periodic wave solutions of the KdV equation and the Boussinesq equation were obtained [10,11]. Such both methods indeed exhibit some advantages. For example, they don't need any Lax pairs and Riemann surface for the considered equation, allow the explicit construction of multi-periodic wave solutions, only rely on the existence of the Hirota's bilinear form, as well as all parameters appearing in Riemann matrix are arbitrary. There is a key difference between Chow's method and Nakamura's method, Chow's method can obtain rational solutions in term of theta function which is, however, not multi-periodic. Nakamura's method can obtain multi-periodic wave solutions interm of theta function which, however, is not rational form. Recently, further development of Nakamura's method was made to investigate the discrete Toda lattice, $(2+1)$-dimensional Kadomtsev–Petviashvili equation, Bogoyavlenskii's breaking soliton equation and Hirota equation [10–17]. However, some repetitive recursion and computation must be preformed in the construction of periodic wave solution for each equation.

The motivation of this Letter is to considerably improves the key steps of the above existing methods. We propose some theta function bilinear formulae, which actually provide us a direct and unifying way for applying in a class of nonlinear differential and difference equations. Once a nonlinear equation is written in bilinear forms, then the periodic wave solutions of the nonlinear equation can be obtained directly by using the formula. Moreover, we propose a simple and effective method to analyze asymptotic properties of the periodic solutions. As illustrative examples, we consider $(2+1)$-dimensional modified Bogoyavlenskii–Schiff equation and differential–difference KdV equation, whose periodic wave solutions seem not available to the knowledge of the authors.

The organization of this Letter is as follows. In Section 2, we briefly introduce a Hirota bilinear operator and a Riemann theta function. In particular, we provide a key formula for constructing periodic wave solutions for both differential and difference equations. As applications of our method, in Sections 3 and 4, we construct double periodic wave solutions to the $(2+1)$-dimensional modified Bogoyavlenskii–Schiff equation and differential–difference KdV equation, respectively. In addition, it is rigorously shown that the double periodic wave solutions tend to the soliton solutions under small amplitude limits.

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2010.07.005

2. Hirota bilinear operator and Riemann theta function

To fix the notations we recall briefly some notions that will be used in this Letter. The Hirota bilinear operators D_x, D_t and D_n are defined as follows:

$$D_x^m D_t^k f(x,t)\cdot g(x,t) = (\partial_x-\partial_{x'})^m(\partial_t-\partial_{t'})^k f(x,t)g(x',t')\big|_{x'=x,t'=t},$$
$$e^{\delta D_n} f(n)\cdot g(n) = e^{\delta(\partial_n-\partial_n')} f(n)g(n')\big|_{n'=n} = f(n+\delta)g(n-\delta),$$
$$\cosh(\delta D_n) f(n)\cdot g(n) = \frac{1}{2}\left(e^{\delta D_n}+e^{-\delta D_n}\right) f(n)\cdot g(n),$$
$$\sinh(\delta D_n) f(n)\cdot g(n) = \frac{1}{2}\left(e^{\delta D_n}-e^{-\delta D_n}\right) f(n)\cdot g(n).$$

Proposition 1. *The Hirota bilinear operators D_x, D_t and D_n have properties* [1–5]

$$D_x^m D_t^k e^{\xi_1}\cdot e^{\xi_2} = (\alpha_1-\alpha_2)^m(\omega_1-\omega_2)^k e^{\xi_1+\xi_2},$$
$$e^{\delta D_n} e^{\xi_1}\cdot e^{\xi_2} = e^{\delta(\nu_1-\nu_2)} e^{\xi_1+\xi_2},$$
$$\cosh(\delta D_n) e^{\xi_1}\cdot e^{\xi_2} = \cosh\left[\delta(\nu_1-\nu_2)\right] e^{\xi_1+\xi_2},$$
$$\sinh(\delta D_n) e^{\xi_1}\cdot e^{\xi_2} = \sinh\left[\delta(\nu_1-\nu_2)\right] e^{\xi_1+\xi_2},$$

where $\xi_j = \alpha_j x+\omega_j t+\nu_j n+\sigma_j$, and α_j, ω_j, ν_j, σ_j, $j=1,2$ are parameters and $n\in\mathbb{Z}$ is a discrete variable. More generally, we have

$$F(D_x,D_t,D_n)e^{\xi_1}\cdot e^{\xi_2} = F\left(\alpha_1-\alpha_2,\omega_1-\omega_2,\exp\left[\delta(\nu_1-\nu_2)\right]\right)e^{\xi_1+\xi_2}, \tag{2.1}$$

where $F(D_x,D_t,D_n)$ is a polynomial about operators D_x, D_t and D_n. This properties are useful in deriving Hirota's bilinear form and constructing periodic wave solutions of nonlinear equations.

In the following, we introduce a general Riemann theta function and discuss its periodicity, which plays a central role in the construction of periodic solutions of nonlinear equations. The Riemann theta function reads

$$\vartheta\begin{bmatrix}\varepsilon\\ s\end{bmatrix}(\xi,\tau) = \sum_{m\in\mathbb{Z}}\exp\left\{2\pi i(\xi+\varepsilon)(m+s)-\pi\tau(m+s)^2\right\}. \tag{2.2}$$

Here the integer value $m\in\mathbb{Z}$, complex parameter $s,\varepsilon\in\mathcal{C}$, and complex phase variables $\xi\in\mathcal{C}$; The $\tau>0$ which is called the period matrix of the Riemann theta function.

In the definition of the theta function (2.2), for the case $s=\varepsilon=0$, hereafter we use $\vartheta(\xi,\tau)=\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](\xi,\tau)$ for simplicity. Moreover, we have $\vartheta\left[\begin{smallmatrix}\varepsilon\\0\end{smallmatrix}\right](\xi,\tau)=\vartheta(\xi+\varepsilon,\tau)$.

Definition 1. A function $g(t)$ on $\mathbb{C}$ is said to be quasi-periodic in t with fundamental periods $T_1,\dots,T_k\in\mathbb{C}$, if $T_1,\dots,T_k$ are linearly dependent over $\mathbb{Z}$ and there exists a function $G(y_1,\dots,y_k)$, such that

$$G(y_1,\dots,y_j+T_j,\dots,y_k) = G(y_1,\dots,y_j,\dots,y_k),\quad \text{for all } y_j\in\mathbb{C},\ j=1,\dots,k,$$
$$G(t,\dots,t,\dots,t) = g(t).$$

In particular, $g(t)$ is called double periodic as $k=2$, and it becomes periodic if and only if $T_j=m_jT$, $j=1,\dots,k$.

Let's first see the periodicity of the theta function $\vartheta(\xi,\tau)$.

Proposition 2. *(See [18,19].) The theta function $\vartheta(\xi,\tau)$ has the periodic properties*

$$\vartheta(\xi+1+i\tau,\tau) = \exp(-2\pi i\xi+\pi\tau)\vartheta(\xi,\tau). \tag{2.3}$$

We regard the vectors 1 and $i\tau$ as periods of the theta function $\vartheta(\xi,\tau)$ with multipliers 1 and $\exp(-2\pi i\xi+\pi\tau)$, respectively. Here, $i\tau$ is not a period of theta function $\vartheta(\xi,\tau)$, but it is the period of the functions $\partial_\xi^2\ln\vartheta(\xi,\tau)$, $\partial_\xi\ln[\vartheta(\xi+e,\tau)/\vartheta(\xi+h,\tau)]$ and $\vartheta(\xi+e,\tau)\vartheta(\xi-e,\tau)/\vartheta(\xi+h,\tau)^2$.

Proposition 3. *The meromorphic functions $f(\xi)$ on $\mathbb{C}$ are as follow*

(i) $f(\xi)=\partial_\xi^2\ln\vartheta(\xi,\tau)$, $\xi\in\mathbb{C}$,
(ii) $f(\xi)=\partial_\xi\ln\frac{\vartheta(\xi+e,\tau)}{\vartheta(\xi+h,\tau)}$, $\xi,e,h\in\mathbb{C}$,
(iii) $f(\xi)=\frac{\vartheta(\xi+e,\tau)\vartheta(\xi-e,\tau)}{\vartheta(\xi,\tau)^2}$, $\xi,e,h\in\mathbb{C}$,

then in all three cases (i)–(iii), *it holds that*

$$f(\xi+1+i\tau) = f(\xi),\quad \xi\in\mathbb{C}, \tag{2.4}$$

that is, $f(\xi)$ is a double periodic function with 1 and $i\tau$.

Proof. By using (2.3), it is easy to see that

$$\frac{\partial_\xi\vartheta(\xi+1+i\tau,\tau)}{\vartheta(\xi+1+i\tau,\tau)} = -2\pi i+\frac{\partial_\xi\vartheta(\xi,\tau)}{\vartheta(\xi,\tau)},$$

or equivalently

$$\partial_\xi\ln\vartheta(\xi+1+i\tau,\tau) = -2\pi i+\partial_\xi\ln\vartheta(\xi,\tau). \tag{2.5}$$

Differentiating (2.5) with respective to ξ again immediately proves the formula (2.4) for the case (i). The formula (2.4) can be proved for the cases (ii) and (iii) in a similar manner. □

Theorem 1. *Suppose that $\vartheta\left[\begin{smallmatrix}\varepsilon'\\0\end{smallmatrix}\right](\xi,\tau)$ and $\vartheta\left[\begin{smallmatrix}\varepsilon\\0\end{smallmatrix}\right](\xi,\tau)$ are two Riemann theta functions, in which $\xi=\alpha x+\omega t+\nu n+\sigma$. Then Hirota bilinear operators D_x, D_t and D_n exhibit the following perfect properties when they act on a pair of theta functions*

$$D_x\vartheta\begin{bmatrix}\varepsilon'\\0\end{bmatrix}(\xi,\tau)\cdot\vartheta\begin{bmatrix}\varepsilon\\0\end{bmatrix}(\xi,\tau) = \sum_{\mu=0,1}\partial_x\vartheta\begin{bmatrix}\varepsilon'-\varepsilon\\-\mu/2\end{bmatrix}(2\xi,2\tau)\Big|_{\xi=0}\vartheta\begin{bmatrix}\varepsilon'+\varepsilon\\ \mu/2\end{bmatrix}(2\xi,2\tau), \tag{2.6}$$

$$\exp(\delta D_n)\vartheta\begin{bmatrix}\varepsilon'\\0\end{bmatrix}(\xi,\tau)\cdot\vartheta\begin{bmatrix}\varepsilon\\0\end{bmatrix}(\xi,\tau) = \sum_{\mu=0,1}\exp(\delta D_n)\vartheta\begin{bmatrix}\varepsilon'-\varepsilon\\-\mu/2\end{bmatrix}(2\xi,2\tau)\Big|_{\xi=0}\vartheta\begin{bmatrix}\varepsilon'+\varepsilon\\ \mu/2\end{bmatrix}(2\xi,2\tau), \tag{2.7}$$

where the notation $\sum_{\mu=0,1}$ represents two different transformations corresponding to $\mu=0,1$. The bilinear formula for t is the same as (2.6) by replacing ∂_x with ∂_t.

In general, for a polynomial operator $F(D_x,D_t,D_n)$ with respect to D_x, D_t and D_n, we have the following useful formula

$$F(D_x,D_t,D_n)\vartheta\begin{bmatrix}\varepsilon'\\0\end{bmatrix}(\xi,\tau)\cdot\vartheta\begin{bmatrix}\varepsilon\\0\end{bmatrix}(\xi,\tau) = \sum_{\mu}C(\varepsilon',\varepsilon,\mu)\vartheta\begin{bmatrix}\varepsilon'+\varepsilon\\ \mu/2\end{bmatrix}(2\xi,2\tau), \tag{2.8}$$

in which, explicitly