Physics Letters A 378 (2014) 480–483

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Derivation of invariant varieties of periodic points from singularity confinement in the case of Toda map

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ARTICLE INFO

Article history:
Received 17 April 2012
Received in revised form 28 June 2013
Accepted 3 December 2013
Available online 11 December 2013
Communicated by A.P. Fordy

Keywords:
Integrable maps
Singularity confinement
Invariant variety of periodic points

ABSTRACT

We have shown in [1] that the invariant varieties of periodic points (IVPP) of all periods of some higher dimensional rational maps can be derived, iteratively, from the singularity confinement (SC). We generalize this algorithm, in this paper, to apply to any birational map, which has more invariants than the half of the dimension.

1. Introduction

Let us consider a map

$$F : x = (x_1, x_2, \dots, x_d) \to X = (X_1, X_2, \dots, X_d), \quad x, X \in \hat{\mathbb{C}}^d \tag{1}$$

where $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cup \{0/0\}$. Here $\{0/0\}$ denotes a set of indeterminate points, which itself should be clarified in the study. Since we consider rational maps it is convenient to write them as

$$X_j = \frac{N_j(x)}{D_j(x)}, \quad j = 1, \dots, d, \tag{2}$$

where $N_j(x),\ D_j(x) \in \mathbb{C}[x]$ are irreducible polynomials. We assume that the map is birational.

Let j be one of $\{1, 2, \dots, d\}$ and denote by Σ_j the variety of zero set of $D_j(x)$. We also denote $\Sigma^+ := \bigcup_j \Sigma_j$. The points on Σ^+ are mapped to $\Lambda(\infty) := F(\Sigma^+)$, which are divergent. But, unless a point of $\Lambda(\infty)$ is a fixed point of the map, there is a possibility that it returns to a finite point after some steps of the map. This is the phenomenon known as singularity confinement (SC) [2,3]. If the points return to a finite region after m_{sc} iteration of the map, we call this number m_{sc}, the 'steps of SC'. This means that none of $D_j^{(m_{sc}+1)}(\Sigma_j)$, $j = 1, 2, \dots, d$, in $F^{(m_{sc}+1)}(\Sigma_j)$ is identically zero, while $F^{(m_{sc})}(\Sigma_j)$ is divergent. It is not difficult to see how this phenomenon takes place [1]. If Σ^- is the zero set of the denominators of the inverse map F^{-1}, it is mapped to $F^{-1}(\Sigma^-) \in \Lambda(\infty)$ by the inverse map. Conversely the points on $\Lambda(\infty)$ are mapped back to Σ^- by the forward map F. From this it is clear that when $F^{(m_{sc})}(\Sigma_j) \in F^{-1}(\Sigma^-)$, it is mapped to $F^{(m_{sc}+1)}(\Sigma_j) \in \Sigma^-$, which is finite:

$$\begin{array}{ccccccccccccc} \Sigma_j & \to & F(\Sigma_j) & \to & F^{(2)}(\Sigma_j) & \to & \cdots & \to & F^{(m_{sc})}(\Sigma_j) & \to & F^{(m_{sc}+1)}(\Sigma_j) & \to & \cdots \\ \cap & & \cap & & \cap & & & & \cap & & \cap & & \\ \Sigma^+ & \to & \Lambda(\infty) & \to & \Lambda(\infty) & \to & \cdots & \to & F^{(-1)}(\Sigma^-) & \to & \Sigma^- & \to & \cdots \end{array} \tag{3}$$

This is the mechanism that the SC phenomenon undergoes. We should mention here that, although we assumed that m_{sc} is common to all Σ_j's, this may not be true in general. Nevertheless all examples we discuss in this paper will satisfy this condition.

Now we assume that the map has p invariants $\{H_1(x), H_2(x), \dots, H_p(x)\}$. Since the map is constrained on the level set

$$V(h) = \{x \mid H_1(x) = h_1, H_2(x) = h_2, \dots, H_p(x) = h_p\}$$

with h_i's being some constants, not all the periodicity conditions $F^{(n)}(x) = x$ of period n, but only $d - p$ of them are independent, which we write as

$$\Gamma_j^{(n)}(\xi, h) = 0, \quad j = 1, 2, \dots, d - p,\ n \geqslant 2. \tag{4}$$

Here $\xi = \{\xi_1, \xi_2, \dots, \xi_{d-p}\}$ is the coordinate which parameterizes the level set $V(h)$. In general (4) will fix ξ for all $h = \{h_1, h_2, \dots, h_p\}$, so that points of period n form a set of discrete points. We notice that the fixed points of the map are not counted as periodic points in (4) and hereafter.

But it might happen that some of the conditions (4) determine relations among the invariants instead of fixing all ξ. If $p \geqslant d/2$,

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0375-9601/$ – see front matter
http://dx.doi.org/10.1016/j.physleta.2013.12.007

in particular, it is possible that all conditions (4) of some period k determine relations among the invariants h, and do not fix ξ at all. In this case the conditions (4), which we write as

$$\gamma_j^{(k)}(h) = 0, \quad j = 1, 2, \dots, d - p, \tag{5}$$

specify the geometry of the level set $V(h)$. We notice that all points on the variety defined by

$$v^{(k)} = \{x \mid \gamma_j^{(k)}(H(x)) = 0,\ j = 1, 2, \dots, d - p\} \tag{6}$$

are points of period k. We call $v^{(k)}$ an invariant variety of periodic points, or IVPP, of period k. We have studied in [4] many rational integrable maps and found that all their periodic points form IVPPs, as far as $p \geqslant d/2$ is satisfied.

We can prove the following theorem [1,4].

IVPP theorem. *When $p \geqslant d/2$, an IVPP and a set of discrete points of any period cannot exist simultaneously in one map.*

Let us present here an outline of the proof, but leaving the details to Ref. [1]. Suppose the points of period k are on the IVPP $v^{(k)}$ of (6) and the points of period n ($\neq k$) satisfy (4) and form a discrete set of points. Since h is free in $\Gamma_j^{(n)}(\xi, h)$ we can always choose the level set to satisfy $\{\gamma_j^{(k)}(h) = 0\}$. It means that the discrete set of points of period n are on the variety $v^{(k)}$, which are totally occupied by points of period k. This contradicts to our assumption $n \neq k$.

Another important observation in [1] is the fact that IVPPs of all periods can be derived iteratively once the map is recovered from the SC. It was shown explicitly by studying the 3 dimensional Lotka–Volterra map (3dLV),

$$F(x) = \left(x_1 \frac{1 - x_2 + x_2 x_3}{1 - x_3 + x_3 x_1},\ x_2 \frac{1 - x_3 + x_3 x_1}{1 - x_1 + x_1 x_2},\ x_3 \frac{1 - x_1 + x_1 x_2}{1 - x_2 + x_2 x_3}\right), \tag{7}$$

which has two invariants

$$f = x_1 x_2 x_3 - (1 - x_1)(1 - x_2)(1 - x_3),$$
$$g = 1 + (1 - x_1)(1 - x_2)(1 - x_3).$$

A point $p^{(0)} = (x_1, x_2, \frac{1}{1-x_1}) \in \Sigma_1 \subset \Sigma^+$, which satisfies $D_1(p^{(0)}) = 0$ is mapped iteratively according to

$$p^{(0)} \to (\infty, 0, 1) \to (1, 0, \infty) \to \left(\frac{1}{1 - x_1}, x_2, x_1\right) \in \Sigma^-, \tag{8}$$

hence the steps of SC is $m_{sc} = 3$.

We notice that, in terms of the invariants, we can write $p^{(0)}$ as

$$p^{(0)} = \left(\frac{f}{f + g}, \frac{g(f + g - 1)}{f}, \frac{f + g}{g}\right).$$

Therefore it is apparent that all $p^{(n)}$'s must be written only by the invariants. In fact we find

$$p^{(3)} = \left(\frac{f + g}{g}, \frac{g(f + g - 1)}{f}, \frac{f}{f + g}\right).$$

From this expression we see that the denominator $D_1^{(3)}$ of $X_1^{(3)}(p^{(0)})$ is given by g. Because $X^{(1)}(p^{(0)})$ is divergent, g must vanish at the points of period 2. In other words the set of points satisfying $g(x) = 0$ form an IVPP of period 2. We can continue this procedure to obtain all IVPPs of the 3dLV,

$$\gamma^{(2)} = g, \qquad \gamma^{(3)} = f^2 + fg + g^2,$$
$$\gamma^{(4)} = f^3 + (1 - g)(f + g)^3, \quad \dots$$

which agree exactly with those derived from the periodicity conditions (4) directly [4].

We can apply this algorithm to other maps when the number p of the invariants is $d - 1$. In fact we can derive IVPPs of 3d KdV map and 4dLV map in this method. When p is less than $d - 1$, a single polynomial of the invariants is not sufficient to determine IVPP of each period. Since we have not studied such cases so far, we must develop a new method. For this purpose we propose in Section 2 a new algorithm and show in Section 3 how it works when $p = d - 2$, including the 3 point Toda map.

2. Algorithm for general cases

The key point of the method used in the derivation of IVPPs of the 3dLV map was to parameterize the zero set of the denominators of the map in terms of the invariants. We now ask if there exists a way to derive IVPPs by the SC when $d/2 \leqslant p \leqslant d - 2$. To answer this question we notice that, in addition to $p^{(0)} \in \Sigma^+$, we need $d - p - 1$ other conditions to write down $p^{(0)}$ by the invariants. We would like to propose, in this section, a set of such additional conditions and provide an algorithm which enables us to generate all IVPPs from the SC in general cases.

To this end let F be the rational map of (1) with p invariants. We assume that $p \geqslant d/2$ and m_{sc} is finite. Without loss of generality we assume $p^{(0)} \in \Sigma_1$, or equivalently $D_1(p^{(0)}) = 0$. Now we propose the following algorithm:

Algorithm

1. Determination of the initial point $p^{(0)}$:
   We impose additional $d - p - 1$ conditions $D_1^{(k)}(p^{(0)}) = 0$, $k = 2, 3, \dots, d - p$ and solve
   $$\{D_1^{(1)}(x) = 0, D_1^{(2)}(x) = 0, \dots, D_1^{(d-p)}(x) = 0,$$
   $$H_1(x) = h_1, H_2(x) = h_2, \dots, H_p(x) = h_p\} \tag{9}$$
   for x to determine $p^{(0)}$ by the invariants $h = (h_1, h_2, \dots, h_p)$.
2. Generation of IVPPs:
   Compute $F^{(k)}(p^{(0)})$ and get
   $$\{D_1^{(k+1)}(p^{(0)}), D_1^{(k+2)}(p^{(0)}), \dots, D_1^{(k+d-p)}(p^{(0)})\},$$
   $$k \geqslant m_{sc} - 1,$$
   from which we find $d - p$ irreducible polynomials of h, one from each element. They are nothing but $\{\gamma_j^{(k)}(h)\}$ of (5).

In order to justify our Algorithm we first notice that $D_1^{(2)}(p^{(0)})$ is not identically zero when $D_1^{(1)}(p^{(0)}) = 0$, because, otherwise, $D_1^{(n)}(p^{(0)}) = 0$ for all $n \geqslant 2$, and contradicts to our assumption that m_{sc} is finite. Therefore the conditions (9) can determine $p^{(0)}$. The second part of the algorithm is apparent because the k period conditions of the map require $D_1^{(j+k)}(p^{(0)}) = 0$ if $D_1^{(j)}(p^{(0)}) = 0$ for all j.

3. Application to some maps

We have already derived the IVPP of period 3 of the 3 point Toda map in [4]. We would like to mention, however, that the analysis using computer algebra becomes much harder as the degrees of freedom of the map increases, if we derive the IVPPs directly from the periodicity conditions of the map. We were not able to find IVPPs of periods higher than 3 by using our personal computer.

We apply, in this section, our algorithm in the cases of 4dLV, 5dLV and 3 point Toda map. The 3 point Toda map is related to