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Discrete chaos in fractional sine and standard maps

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ABSTRACT

Fractional standard and sine maps are proposed by using the discrete fractional calculus. The chaos behaviors are then numerically discussed when the difference order is a fractional one. The bifurcation diagrams and the phase portraits are presented, respectively.

1. Introduction

As a reliable tool for mathematical modeling, the fractional calculus has been extensively used in a large range of physical phenomena and gained much fruitful results in the past decades [1–5]. In the nature, social and computer science, a lot of discrete nonlinear problems and the discrete dynamics behaviors possess long-range interaction traits. Researchers are frequently undertaking to develop the methods and theories from the fractional calculus to the discrete cases. Some efforts [6–10] have been made in this topic.

In the frame of the time scale theory [11], the discrete fractional calculus (DFC) [12–14] was proposed to describe the dynamics of the discrete time. It was pointed out that the DFC is the development of the theory of the fractional calculus on time scales [12]. In view of this point, some other works have been done, such as the Taylor series [15], the definitions of the fractional differences and their properties [16,17], the Laplace transform [18] and the existence results [19,20]. However, less work was contributed to the aspects of the dynamics behaviors.

In order to deeply understand the background of the discrete dynamics behaviors, our main objective is to introduce applications of the discrete fractional calculus on an arbitrary time scale [12–14] and use the theories of delta difference equations to reveal the discrete chaos behaviors of the fractionalized standard map. The Letter is organized as follows: Section 2 introduces the definitions and the properties of the DFC; Section 3 presents fractional sine map and standard maps on time scales; From the discrete integral expression, Section 4 gives the discrete chaotical solutions and the phase portraits of the maps while the difference orders while the coefficients are changing.

2. Preliminaries

Let's firstly revisit briefly the definitions of the fractional calculus [1–5].

Definition 2.1. Let $f(t)$ be a function of class $\mathfrak{C}$, i.e. piecewise continuous on $(t_0, +\infty)$ and integrable on any finite subinterval of $(t_0, +\infty)$. Then for $t > 0$, the Riemann–Liouville integral of $f(t)$ of α order is defined as

$$ {}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau \tag{1}$$

where α is a positive real number and $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. Let α be a positive real number, $m-1 < \alpha \leqslant m$, $m \in \mathbb{N}^+$, and $f^{(m)}(t)$ exist and be a function of class $\mathfrak{C}$. Then the Caputo fractional derivative of $f(t)$ of order α is defined as

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$$ {}_0^C D_t^\alpha f(t) = {}_0 I_t^{m-\alpha} f^{(m)}(t), \quad t > 0. \tag{2} $$

We can mention the above definitions as the continuous fractional calculus. For the Caputo derivative of the power function t^μ, $\mu > 0$, if $0 \leqslant m-1 < \alpha \leqslant m < \mu + 1$, then we have

$$ {}_0^C D_t^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \quad t > 0. \tag{3} $$

Considering the DFC, the defined function $f(t)$ is changed as a sequence $f(n)$. Let $\mathbb{N}_a$ denotes the isolated time scale and $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ ($a \in \mathbb{R}$ fixed). The difference operator Δ is defined as $\Delta f(n) = f(n+1) - f(n)$.

Definition 2.3. (See [12].) Let u: $\mathbb{N}_a \to \mathbb{R}$ and $0 < \nu$ be given. Then the fractional sum of ν order is defined by

$$ \Delta_a^{-\nu} u(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \big(t - \sigma(s)\big)^{(\nu-1)} u(s), \quad t \in \mathbb{N}_{a+\nu} \tag{4} $$

where a is the starting point, $\sigma(s) = s+1$ and $t^{(\nu)}$ is the falling function defined as

$$ t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}. \tag{5} $$

Definition 2.4. (See [16].) For $0 < \nu$, $\nu \notin \mathbb{N}$ and $u(t)$ defined on $\mathbb{N}_a$, the Caputo-like delta difference is defined by

$$ \begin{aligned} {}^C\Delta_a^\nu u(t) &:= \Delta_a^{-(m-\nu)} \Delta^m u(t) \\ &= \frac{1}{\Gamma(m-\nu)} \sum_{s=a}^{t-(m-\nu)} \big(t - \sigma(s)\big)^{(m-\nu-1)} u(s), \\ & t \in \mathbb{N}_{a+m-\nu}, \quad m = [\nu] + 1, \end{aligned} \tag{6} $$

where ν is the difference order.

Theorem 2.5. *(See [19].) For the delta fractional difference equation*

$$ \begin{aligned} & {}^C\Delta_a^\nu u(t) = f\big(t+\nu-1, u(t+\nu-1)\big), \\ & \Delta^k u(a) = u_k, \quad m = [\nu]+1, \quad k = 0, \ldots, m-1, \end{aligned} \tag{7} $$

the equivalent discrete integral equation can be obtained as

$$ \begin{aligned} u(t) = u_0(t) + \frac{1}{\Gamma(\nu)} \sum_{s=a+m-\nu}^{t-\nu} \big(t-\sigma(s)\big)^{(\nu-1)} \\ \times f\big(s+\nu-1, u(s+\nu-1)\big), \quad t \in \mathbb{N}_{a+m} \end{aligned} \tag{8} $$

where the initial iteration reads

$$ u_0(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k u(a). \tag{9} $$

The complex difference equation with long-term memory is obtained here. It can reduce to the classical one for the difference order $\nu = 1$ but the integer one doesn't hold the discrete memory. From Eq. (6) to Eq. (8), the domain is changed from $\mathbb{N}_{a+m-\nu}$ to $\mathbb{N}_{a+m}$ and the function $u(t)$ is preserved to define on the isolated time scale $\mathbb{N}_a$ in the fractional sums. We can see that the discrete fractional calculus is a crucial tool in the initialization of the fractional difference equations.

3. Fractional sine and standard maps

Directly from the fractional calculus, Tarasov [8] investigated the maps derived from the fractional differential equations and discussed the chaotical behaviors of the fractional standard map. In this Letter, we suggest the application of the DFC to fractional generalizations of the discrete maps. For example, consider the one dimensional sine map

$$ x_{n+1} = x_n + \mu \sin(x_n) \tag{10} $$

where μ is the amplitude of the pulses in the motion of equation.
Eq. (10) can be rewritten as

$$ \Delta x(n) = \mu \sin\big(x(n)\big). \tag{11} $$

From the discrete fractional calculus, the fractional one can be given as

$$ {}^C\Delta_a^\nu x(t) = \mu \sin\big(x(t+\nu-1)\big), \quad 0 < \nu < 1, \quad t \in \mathbb{N}_{a+1-\nu} \tag{12} $$

where ν is the difference order.

The two dimensional standard map reads

$$ \begin{cases} x_{n+1} = x_n - K \sin(y_n), \\ y_{n+1} = y_n + x_{n+1}. \end{cases} \tag{13} $$

The map was studied by Chirikov in 1979 [21]. $x(n)$ and $y(n)$ are the momentum and coordinate, respectively. They are taken modulo 2π. The map describes the dynamics of the kicked rotor. Considering the fractional generalization of the momentum $x(n)$, we modify the standard map as a fractional one

$$ \begin{cases} {}^C\Delta_a^\nu x(t) = -K \sin\big(y(t+\nu-1)\big), \\ \quad 0 < \nu \leqslant 1, \quad t \in \mathbb{N}_{a+1-\nu}, \\ y(n) = y(n-1) + x(n). \end{cases} \tag{14} $$

4. Chaos in the discrete fractional maps

From Theorem 2.5, we can obtain the following equivalent discrete integral form for $0 < \nu < 1$

$$ \begin{aligned} u(t) = u(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} \big(t-\sigma(s)\big)^{(\nu-1)} \\ \times f\big(s+\nu-1, u(s+\nu-1)\big), \quad t \in \mathbb{N}_{a+1} \end{aligned} \tag{15} $$

where $\frac{(t-\sigma(s))^{(\nu-1)}}{\Gamma(\nu)}$ is a discrete kernel function and $(t-\sigma(s))^{(\nu-1)} = \frac{\Gamma(t-s)}{\Gamma(t-s+1-\nu)}$. As a result, the numerical formula can be presented explicitly

$$ u(n) = u(a) + \frac{1}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} f\big(j-1, u(j-1)\big). \tag{16} $$

For the sine map (12), an explicit numerical formula can be given as

$$ x(n) = x(a) + \frac{\mu}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \sin\big(x(j-1)\big). \tag{17} $$

Let $\nu = 1$, $a = 0$, $x(0) = 0.3$, $n = 200$ and the μ be fixed. In what follows, Fig. 1 is the bifurcation diagram where the step size of the μ is set as 0.01. Fig. 2 is the same bifurcation diagram except the difference order $\nu = 0.8$. We can observe that the chaotic zones are clearly dependent on the changing difference order ν.

For the fractional standard map (14), we can have the numerical formula