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Conservation laws and normal forms of evolution equations

Roman O. Popovych [a,b], Artur Sergyeyev [c,*]

[a] Fakultät für Mathematik, Universität Wien, Nordbergstraße 15, A-1090 Wien, Austria
[b] Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., Kyiv-4, Ukraine
[c] Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 746 01 Opava, Czech Republic

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ABSTRACT

We study local conservation laws for evolution equations in two independent variables. In particular, we present normal forms for the equations admitting one or two low-order conservation laws. Examples include Harry Dym equation, Korteweg–de Vries-type equations, and Schwarzian KdV equation. It is also shown that for linear evolution equations all their conservation laws are (modulo trivial conserved vectors) at most quadratic in the dependent variable and its derivatives.

1. Introduction

The role played in the sciences by linear and nonlinear evolution equations and, in particular, by conservation laws thereof, is hard to overestimate (recall e.g. linear and nonlinear Schrödinger equations and the Korteweg–de Vries (KdV) equation in physics, reaction–diffusion systems in chemistry and biology, and the Black–Scholes equation in the finance, to name just a few). For instance, the discovery of higher conservation laws for the KdV equations provided an important milestone on the way that has eventually lead to the discovery of the inverse scattering transform and the modern theory of integrable systems, see e.g. [21, 22]. However, the theory of conservation laws for evolution equations is still far from being complete even for the simplest case of two independent variables, and in the present Letter we address some issues of the theory in question for this very case.

We shall deal with an evolution equation in two independent variables,

$$u_t = F(t, x, u_0, u_1, \dots, u_n), \quad n \geqslant 2, \qquad F_{u_n} \neq 0, \tag{1}$$

where $u_j \equiv \partial^j u/\partial x^j$, $u_0 \equiv u$, and $F_{u_j} = \partial F/\partial u_j$. We shall also employ, depending on convenience or necessity, the following notation for low-order derivatives: $u_x = u_1$, $u_{xx} = u_2$, and $u_{xxx} = u_3$.

There is a considerable body of results on conservation laws of evolution equations of the form (1). For instance, in the seminal paper [8] the authors studied, *inter alia*, conservation laws of Eq. (1) with $\partial F/\partial t = 0$ for $n = 2$. They proved that the possible dimensions of spaces of inequivalent conservation laws for such equations are 0, 1, 2 and ∞, and described the equations possessing spaces of conservation laws of these dimensions (the precise definitions of equivalence and order of conservation laws are given in the next section). These results were further generalized in [28] for the case when F explicitly depends on t.

Important results on conservation laws of (1), typically under the assumptions of polynomiality and t, x-independence of F and of the conservation laws themselves, were obtained in [1–4,10–12, 15]. However, for general equation (1) there is no simple picture analogous to that of the second-order case discussed above. For instance, unlike the second-order case, there exist odd-order evolution equations that possess infinitely many inequivalent conservation laws of increasing orders without being linearizable. Rather, such equations are integrable via the inverse scattering transform, the famous KdV equation providing a prime example of such behavior, see e.g. [13,15,21] and references therein; for the fifth-order equations see [9].

Note that many results on symmetries and conservation laws were obtained using the formal symmetry approach and modifications thereof, see e.g. the recent survey [20] and references therein, in particular [19,34]. For instance, it was shown that an equation of the form (1) of even order ($n = 2m$) has no conservation laws (modulo trivial ones) of order greater than m, see [1,10,13,14] for

* Corresponding author.
E-mail addresses: rop@imath.kiev.ua (R.O. Popovych), Artur.Sergyeyev@math.slu.cz (A. Sergyeyev).

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details. There also exists a closely related approach to the study of symmetries and conservation laws of evolution equations, the so-called symbolic method, see [18,29–31] and references therein for details.

However, many important questions concerning the conservation laws of evolution equations were not answered so far. For example, we are not aware of any significant advances in the study of normal forms of evolution equations admitting low-order conservation laws considered in [8,11,28]. In the present Letter we provide such normal forms with respect to contact or point transformations for equations admitting one or two low-order conservation laws, respectively, see Theorem 1 and Theorem 2 below. Let us stress that in what follows we restrict ourselves to considering only *local* conservation laws whose densities and fluxes depend only on the independent and dependent variables and a finite number of the derivatives of the latter.

The complete description of local conservation laws for *local linear* evolution equations with t,x-dependent coefficients was also missing so far. Below we show that linear even-order equations of the form (1) can only possess conservation laws linear in u_j for all $j=0,1,2,\dots$ while the odd-order equations can further admit the conservation laws (at most) quadratic in u_j, see Theorem 3 and Theorem 4, Corollary 6 and Theorem 5 below. This naturally generalizes some earlier results from [3,12]; cf. also [5]. The generation of linear and quadratic conservation laws for linear differential equations is also discussed in some depth in [24, Section 5.3].

Below we denote by $\mathrm{CL}(\mathcal{E})$ the space of local conservation laws of $\mathcal{E}$ (cf. Section 3), where $\mathcal{E}$ denotes a fixed equation from the class (1). In what follows D_t and D_x stand for the total derivatives (see e.g. [24] for details) with respect to the variables t and x,

$$D_t=\partial_t+u_t\partial_u+u_{tt}\partial_{u_t}+u_{tx}\partial_{u_x}+\cdots,$$

$$D_x=\partial_x+u_x\partial_u+u_{tx}\partial_{u_t}+u_{xx}\partial_{u_x}+\cdots.$$

As usual, the subscripts like t, x, u, u_x, etc. stand for the partial derivatives in the respective variables.

2. Admissible transformations of evolution equations

The contact transformations mapping an equation from class (1) into another equation from the same class are well known [17] to have the form

$$\tilde t=T(t),\qquad \tilde x=X(t,x,u,u_x),\qquad \tilde u=U(t,x,u,u_x). \tag{2}$$

The functions T, X and U must satisfy the nondegeneracy assumptions, namely, $T_t\neq 0$ and

$$\operatorname{rank}\begin{pmatrix} X_x & X_u & X_{u_x}\\ U_x & U_u & U_{u_x}\end{pmatrix}=2, \tag{3}$$

and the contact condition

$$(U_x+U_uu_x)X_{u_x}=(X_x+X_uu_x)U_{u_x}. \tag{4}$$

The transformation (2) is uniquely extended to the derivative u_x and to the higher derivatives by the formulas $\tilde u_{\tilde x}=V(t,x,u,u_x)$ and $\tilde u_k\equiv\partial^k\tilde u/\partial\tilde x^k=((1/D_xX)D_x)^kV$, where

$$V=\frac{U_x+U_uu_x}{X_x+X_uu_x}\quad\text{or}\quad V=\frac{U_{u_x}}{X_{u_x}}$$

if $X_x+X_uu_x\neq 0$ or $X_{u_x}\neq 0$, respectively; the possibility of simultaneous vanishing of these two quantities is ruled out by (3).

The transformed equation (1) reads $\tilde u_{\tilde t}=\tilde F$ where

$$\tilde F=\frac{U_u-X_uV}{T_t}F+\frac{U_t-X_tV}{T_t}, \tag{5}$$

and $(X_u,U_u)\neq(0,0)$ because of (3) and (4).

Any transformation of the form (2) leaves the class (1) invariant, and therefore its extension to an arbitrary element F belongs to the contact equivalence group $G^\sim_{\rm c}$ of class (1), so there are no other elements in $G^\sim_{\rm c}$. In other words, the equivalence group $G^\sim_{\rm c}$ generates the whole set of admissible contact transformations in the class (1), i.e., this class is normalized with respect to contact transformations, see [26] for details.

The above results can be summarized as follows.

Proposition 1. *The class of Eqs.* (1) *is contact-normalized. The contact equivalence group $G^\sim_{\rm c}$ of the class* (1) *is formed by the transformations* (2)*, satisfying conditions* (3) *and* (4) *and prolonged to the arbitrary element F by* (5)*.*

Furthermore, the class (1) is also point-normalized. The point equivalence group $G^\sim_{\rm p}$ of this class consists of the transformations of the form

$$\tilde t=T(t),\qquad \tilde x=X(t,x,u),\qquad \tilde u=U(t,x,u),$$
$$\tilde F=\frac{\Delta}{T_tD_xX}F+\frac{U_tD_xX-X_tD_xU}{T_tD_xX}, \tag{6}$$

where T, X and U are arbitrary smooth functions that satisfy the nondegeneracy conditions $T_t\neq 0$ and $\Delta=X_xU_u-X_uU_x\neq 0$.

Notice that there exist subclasses of the class (1) whose sets of admissible contact transformations are *exhausted* by point transformations.

In the present Letter we do not consider more general transformations, e.g., differential substitutions such as the Cole–Hopf transformation.

3. Some basic results on conservation laws

It is well known that for any evolution equation (1) we can assume without loss of generality that the associated quantities like symmetries, cosymmetries, densities, etc., can be taken to be independent of the t-derivatives or mixed derivatives of u. We shall stick to this assumption throughout the rest of the Letter.

Following [24] we shall refer to a (smooth) function of t, x and a finite number of u_j as to a *differential function*. Given a differential function f, its *order* (denoted by $\operatorname{ord} f$) is the greatest integer k such that $f_{u_k}\neq 0$ but $f_{u_j}=0$ for all $j>k$. For $f=f(t,x)$ we assume that $\operatorname{ord} f=0$.

Thus, for a (fixed) evolution equation (1), which we denote by $\mathcal{E}$ as before, we lose no generality [24] in considering only the *conserved vectors* of the form (ρ,σ), where ρ and σ are differential functions which satisfy the condition

$$D_t\rho+D_x\sigma=0\ \operatorname{mod}\ \check{\mathcal{E}}, \tag{7}$$

and $\check{\mathcal{E}}$ means the equation $\mathcal{E}$ together with all its differential consequences. Here ρ is the *density* and σ is the *flux* for the conserved vector (ρ,σ). Let

$$\frac{\delta}{\delta u}=\sum_{i=0}^{\infty}(-D_x)^i\partial_{u_i},$$

$$f_*=\sum_{i=0}^{\infty}f_{u_i}D_x^i,\qquad f_*^\dagger=\sum_{i=0}^{\infty}(-D_x)^i\circ f_{u_i}$$

denote the operator of variational derivative, the Fréchet derivative of a differential function f, and its formal adjoint, respectively. With this notation in mind we readily infer that the condition (7) can be rewritten as $\rho_t+\rho_*F+D_x\sigma=0$. As $\rho_*F=F\delta\rho/\delta u+D_x\zeta$