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The one-dimensional spinless Salpeter Coulomb problem with minimal length

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ABSTRACT

We present an exact analytical treatment of the semi-relativistic spinless Salpeter equation with a one-dimensional Coulomb interaction in the context of quantum mechanics with modified Heisenberg algebra implying the existence of a minimal length. The problem is tackled in the momentum space representation. The bound-state energy equation and the corresponding wave functions are exactly obtained.

1. Introduction

The spinless Salpeter equation (SSE) may be viewed as a natural approximation to the Bethe–Salpeter formalism which constitutes the basic framework for the description of bound states within relativistic quantum field theory. More precisely, the spinless Salpeter equation can be derived from the Bethe–Salpeter equation [1] upon performing the following approximations:

* The elimination of any dependence on time-like variable by assuming a static or instantaneous interactions. This results in the so-called Salpeter equation [2].
* The neglect of any references to the spin degrees of freedom of particles and the restriction to positive energy solutions.

In addition, this equation represents one of the simplest relativistic generalizations of the Schrödinger formalism towards the reconciliation with all the requirements of special relativity through the incorporation of the exact relativistic relation between energy and momentum. This equation is generally used when kinetic relativistic effects cannot be neglected. It is suitable for the description of scalar bosons as well as the spin averaged spectra of bound states of fermions. It appears, for example, in the description of hadrons as bound states of quarks in the context of potential models [3–7].

In the one-particle case, the spinless Salpeter equation takes the form of an eigenvalue problem ($\hbar = c = 1$)

$$H|\psi_k\rangle = E_k|\psi_k\rangle, \quad k = 1, 2, 3, \ldots$$

with E_k the energy eigenvalues corresponding to Hilbert-space eigenvectors $|\psi_k\rangle$, and H the Hamiltonian of the system being of the form

$$H = \sqrt{\mathbf{p}^2 + m^2} + V(\mathbf{r})$$

where $\sqrt{\mathbf{p}^2 + m^2}$ is the relativistic kinetic energy of the particle of mass m and momentum $\mathbf{p}$ and $V(\mathbf{r})$ is an arbitrary position-dependent static interaction potential.

Unfortunately, the presence of the square root renders the spinless Salpeter equation difficult to handle, so that general rigorous results concerning this equation are few. Most of these results have been obtained for the Coulomb potential [8–12]. (For a more detailed history of the spinless Salpeter Coulomb problem we refer the reader to Ref. [13].) Other particular interactions have been also considered in some studies where upper and lower limits on energy levels have been obtained [14–17]. Some more general results also exist.

All the mentioned studies have been performed within the framework of usual quantum mechanics where position and momentum operators acting on the Hilbert space of states verify the standard Heisenberg algebra. They do not take into account the existence of a finite lower bound to the possible resolution of distance, i.e. a minimal observable length. This concept has been recently suggested by several investigations in string theory and quantum gravity [18–22]. From a quantum theoretical point of view, the minimal length may be described as a non-zero minimal uncertainty in position measurements. In [23–26], Kempf et al. showed that this idea may be implemented by introducing

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a specific corrections to the usual canonical commutation relations between position and momentum operators. The resulting uncertainty principle exhibits an intriguing UV/UR mixing. This kind of relation has first appeared in Ads/CFT correspondence [27] and in non-commutative quantum field theory [28]. Physically speaking, the UV/UR mixing means that short distance physics may be probed by long distance physics. This lends an additional justification to the analysis of quantum mechanical problems in the presence of a minimal length. On the other hand, it was also suggested that the minimal length can be related to large extra dimensions [29], to the running coupling constant [30], and to the physics of black hole production [31].

Remark that it has also been argued [25,32] that the formalism of minimal length can provide an effective theory to describe non-pointlike particles such as quasiparticles and various collective excitations in solids, or composite particles, such as nucleons, nuclei and molecules [25]. In this case the minimal length may be viewed as an intrinsic scale characterizing the structure of the considered system and its finite size.

In the last years, a lot of attention has been addressed to the study of the effect of minimal length in quantum mechanical problems [23,24,33–45]. One of the more investigated system in the literature is the Coulomb potential problem [34–40]. This is natural since this system has a crucial role for our understanding of the key points of modern physics. However, as far as we know, no one has reported on the relativistic version of this model. In the present work, we proceed some steps in this direction and we study the one-dimensional spinless Salpeter Coulomb problem under a minimal length assumption. In addition to its importance as a new problem for which the relations giving the bound-states energies can be found exactly, the considered system may have interesting applications in theoretical physics. For example, this potential appears in the investigation of mass spectra of mesons [46], and may also be relevant for the physics of semiconductors and insulators [47].

The rest of the Letter is organized as follows. In Section 2, we introduce the main relations of quantum mechanics with modified Heisenberg algebra. In Section 3, we solve in the momentum space representation the spinless Salpeter equation with a one-dimensional Coulomb interaction in the presence of a minimal length. Then we obtain the exact relations from which the corresponding bound-states energies can be extracted. In Section 4 we give our conclusion.

2. Quantum mechanics with generalized Heisenberg algebra

The modified Heisenberg algebra we shall consider in this Letter is defined by the following commutation relation between the position and momentum operators

$$[\hat{X},\hat{P}]=i\left(1+\beta\hat{P}^2\right) \tag{1}$$

where β is a positive parameter. This deformed commutation relation leads to the generalized uncertainty principle

$$(\Delta X)(\Delta P)\geqslant\frac{1}{2}\left[1+\beta(\Delta p)^2\right] \tag{2}$$

which implies the existence of a non-zero minimal uncertainty in position

$$(\Delta x)_{\min}=\sqrt{\beta}. \tag{3}$$

As mentioned above, the striking feature of (2) is the UV/IR mixing: when ΔP is large (UV), ΔX is proportional to ΔP, and therefore is also large (IR). This phenomena seems to be necessary in dealing with certain types of new physics [28] being considered recently.

A fundamental consequence of the minimal length is the loss of the notion of localization in the position space since space coordinates can no longer be probed with accuracy more than $(\Delta x)_{\min}$. Hence momentum space becomes more convenient in order to solve any eigenvalue problem.

In the momentum space, an explicit representation of the position and momentum operators obeying Eq. (1) is given by

$$\hat{X}=i\left[\left(1+\beta p^2\right)\frac{\partial}{\partial p}+\gamma p\right],\qquad \hat{P}=p \tag{4}$$

where γ is an arbitrary constant. Note that γ does not affect the observable quantities, its choice determines only the weight function in the definition of the scalar product given by

$$\langle\phi|\psi\rangle=\int_{-\infty}^{+\infty}\frac{dp}{(1+\beta p^2)^{1-\frac{\gamma}{\beta}}}\phi^*(p)\psi(p). \tag{5}$$

In the following we will set $\gamma=0$ in order to simplify the calculations.

3. Spinless Salpeter Coulomb problem with minimal length

In this section, we shall solve in the momentum space representation the following eigenvalue problem

$$\left[\sqrt{\hat{P}^2+m^2}+V(\hat{X})\right]|\psi\rangle=E|\psi\rangle \tag{6}$$

with a Coulomb type interaction

$$V(x)=-\frac{\kappa}{x},\quad x\in\mathbf{R}\ (\kappa>0) \tag{7}$$

where position and momentum operators satisfy deformed commutation relation (1). Due to its singularity at the origin, the Coulomb potential is in particular sensitive to whether there is a fundamental minimal length.

Let us note that potential (7) has the same bound states as the hard-core amended Coulomb potential considered in [48]. Indeed, bound states are determined by the potential well of the right half space which is the same for the two interactions and the boundary condition $\psi(x=0)=0$. In our case, this condition follows from the infinite barrier that has the potential in hand in the left half space.

Operating now on the both sides of Eq. (6) with $\hat{X}$ and replacing the position operator by its momentum representation (4) (with $\gamma=0$) we get the following differential equation

$$\left(1+\beta p^2\right)\frac{\partial}{\partial p}\left[\left(E-\sqrt{p^2+m^2}\right)\psi(p)\right]-i\kappa\psi(p)=0. \tag{8}$$

Defining a new function $\varphi(p)$ by

$$\varphi(p)=\left(E-\sqrt{p^2+m^2}\right)\psi(p). \tag{9}$$

Eq. (8) becomes

$$\frac{\partial}{\partial p}\varphi(p)-\frac{i\kappa}{(E-\sqrt{p^2+m^2})(1+\beta p^2)}\varphi(p)=0 \tag{10}$$

the solution of which is given by

$$\varphi(p)=\lambda\exp i\theta(p) \tag{11}$$

where λ is an arbitrary constant and the function $\theta(p)$ is defined as

$$\theta(p)=\int_0^p\frac{\kappa\,du}{(E-\sqrt{u^2+m^2})(1+\beta u^2)}. \tag{12}$$