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A new nonlinear-wave-equation formalism for stimulated Brillioun scattering

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ABSTRACT

A revised formalism of SBS is given based on a new optical nonlinear wave equation which explicitly accounts for the macroscopic spatial inhomogeneity resulting from the induced acoustic wave in the medium. The new equation applies to other scattering phenomena, and more generally to optical wave propagation and interaction in nanostructured media for which characteristic spatial scale lengths of material structures (existing or optically induced) are smaller than the optical wavelength.

In 1964 Chiao, Townes and Stoicheff, [1], discovered stimulated Brillouin scattering (SBS) (in the Russian literature it is known as stimulated Mandelshtam–Brillouin scattering (SMBS)). From its inception SBS has grown to be a substantial part of optical physics with applications over many disciplines of science and technology including spectroscopy, material and laser science, metrology to name but a few. The field has experienced a particularly massive growth in the last decade or so with the development of fiber optic technology. SBS, as originally introduced in [1] and subsequently confirmed experimentally in [2], occurs when an input (pump) optical wave incident on a medium is reflected by an acoustic wave induced in the medium through electrostriction by the interference pattern of that pump and an earlier reflected (and/or scattered) frequency downshifted (Stokes) wave(s). The resulting reflected wave is also frequency downshifted from the pump because of the Doppler-effect arising from movement of the acoustic wave and is therefore also a Stokes wave. Through such interaction the SBS Stokes and acoustic waves are both enhanced by transfer of energy from the pump wave. The wavelength, Λ, of the acoustic wave, which is actually the spatial period of a moving grating of the medium's density, is dependent on the angle of scattering and it ranges from a minimum value of $\sim\lambda/2$ for back-scattering to $\sim\lambda/\theta$ for small angles of scattering, $\theta \ll 1$. Though, as such, the underlying physics of the SBS interaction is clearly understood to involve a medium which is induced to be spatially inhomogeneous, the formal theoretical description of SBS is surprisingly built on the traditional nonlinear wave equation [3,4],

$$\nabla\times\nabla\times\vec{\mathrm{E}}+\frac{1}{c^2}\frac{\partial^2(\varepsilon\vec{\mathrm{E}})}{\partial t^2}=-\frac{4\pi}{c^2}\frac{\partial^2\vec{\mathrm{P}}_{NL}}{\partial t^2}, \tag{1}$$

in which the medium is approximated to be spatially homogeneous. Here $\vec{\mathrm{E}}$ is the optical radiation electric field and $\vec{\mathrm{P}}_{NL}$ is the nonlinear polarization, which is a consequence of the nonlinear change of the dipole moment of an atom or molecule, [3,4], and which is the field source. The necessary condition of the approximation is that the characteristic scale of spatial inhomogeneities in the medium, a, are much less than radiation wavelength, $a \ll \lambda$. Evidently this is not the case for Brillouin scattering, for which the characteristic scale of spatial inhomogeneity is the acoustic wavelength and so $a \leqslant \lambda$ for $\theta \approx \pi$ and $a \geqslant \lambda$ for smaller θ. It is for this reason Eq. (1) cannot be adequate in describing SBS. In this Letter we address this issue. We derive from the classical electrodynamics theory of scattering a revised formalism of SBS based on a new optical nonlinear wave equation which explicitly accounts for macroscopic spatial inhomogeneity resulting from the induced acoustic wave in the medium. Implications of the new formalism to the theory and practice of the SBS and related stimulated scattering interactions will be briefly discussed.

To properly account for the fact that the SBS interaction inherently involves an induced spatial inhomogeneity, which is actually the spatial variation of the medium's permittivity, we must return to the basics of electromagnetic (EM) theory for condensed media. It is shown in [5] that a weak spatial inhomogeneity of the medium's permittivity, $\varepsilon(r)=\varepsilon_0+\Delta\varepsilon(r)$, where $\Delta\varepsilon(r)$ is a tensor, which characterizes the scattering properties of the medium, results in a "spatial" term of kind $\nabla\times\nabla\times[\Delta\varepsilon(r)\bullet\mathrm{E}]$ in the wave equation for the scattered field, $\vec{\mathrm{E}}_S$. Here $[\Delta\varepsilon(r)\bullet\mathrm{E}]$ denotes the vector whose components are $[\Delta\varepsilon_{ik}(r)\mathrm{E}_k]$ with $\Delta\varepsilon_{ik}(r)$ being the components of tensor $\Delta\varepsilon(r)$ which characterizes the scattering properties of the medium for the incident (pump) field, $\vec{\mathrm{E}}=\vec{\mathrm{E}}_p$, and E_k being the components of the pump field. This term accounts for the fact that the scattered radiation field is fed by an incident radiation field, and it is this term which describes traditional Rayleigh scattering and X-ray diffraction. The formal procedures which results in such a term in the linear wave equation

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are given in [5]. Since the nonlinear term on the RHS of Eq. (1) is a consequence of a "spatially local effect", it can contribute to the scattered field independently of the "spatial" term. As such the "spatial" term can be simply added to the nonlinear wave equation (1),

$$\nabla \times \nabla \times \vec{\mathrm{E}}_S + \frac{\varepsilon_0}{c^2}\frac{\partial^2 \vec{\mathrm{E}}_S}{\partial t^2} = -\frac{4\pi}{c^2}\frac{\partial^2 \vec{\mathrm{P}}_{NL}}{\partial t^2} + \frac{1}{\varepsilon_0}\nabla \times \nabla \times \left[\Delta\varepsilon(r) \bullet \mathrm{E}\right]. \tag{2}$$

Since $\nabla \times \nabla \times \vec{\mathrm{K}} = \nabla(\nabla\vec{\mathrm{K}}) - \nabla^2\vec{\mathrm{K}}$ for an arbitrary vector $\vec{\mathrm{K}}$, and taking into account that vector $\vec{\mathrm{E}}_S$ is the normal transverse electric field of the EM wave, for which $\nabla(\nabla\vec{\mathrm{E}}_S) = 0$ and $\nabla \times \nabla \times \vec{\mathrm{E}}_S = -\nabla^2\vec{\mathrm{E}}_S$, the LHS of Eq. (2) can be rewritten to give Eq. (2) in the more familiar form,

$$\nabla^2\vec{\mathrm{E}}_S - \frac{\varepsilon_0}{c^2}\frac{\partial^2 \vec{\mathrm{E}}_S}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 \vec{\mathrm{P}}_{NL}}{\partial t^2} - \frac{1}{\varepsilon_0}\nabla \times \nabla \times \left[\Delta\varepsilon(r) \bullet \mathrm{E}\right]. \tag{3}$$

In the most general case the spatial variation of the permittivity consists of both linear, $\Delta\varepsilon_L$, and nonlinear, $\Delta\varepsilon_{NL}$, contributions and these may also be time dependant,

$$\Delta\varepsilon(r,t) = \Delta\varepsilon_L(r,t) + \Delta\varepsilon_{NL}(r,t). \tag{4}$$

Eq. (3) reduces to the standard nonlinear wave equation for an optically homogeneous medium when both $\Delta\varepsilon_L(r,t)$ and $\Delta\varepsilon_{NL}(r,t)$ are small and/or independent of r. When $\Delta\varepsilon_L(r,t)$ is not small while $\Delta\varepsilon_{NL}(r,t)$ is negligible Eq. (3) describes linear light scattering, including Bragg reflection with or without frequency shift, which accompanies other nonlinear effects described by $\vec{\mathrm{P}}_{NL}$. When $\Delta\varepsilon_{NL}(r,t)$ is not small Eq. (3) describes a range of nonlinearly induced scattering phenomena. Eqs. (3) with (4) is then the generalized form of the nonlinear wave equation for a spatially inhomogeneous nonlinear medium.

As introduced, $\vec{\mathrm{P}}_{NL}$ and $[\Delta\varepsilon(r,t) \bullet \mathrm{E}]$ are independent and can vary in space and time differently. On the other hand they can also be caused by the same kind of nonlinearity. For this case, through replacing $\vec{\mathrm{P}}_{NL}$ by $[\Delta\varepsilon(r,t) \bullet \mathrm{E}]/4\pi$ we obtain the nonlinear wave equation for a spatially inhomogeneous nonlinear medium in the form,

$$\nabla^2\vec{\mathrm{E}}_S - \frac{\varepsilon_0}{c^2}\frac{\partial^2 \vec{\mathrm{E}}_S}{\partial t^2} = \frac{1}{\varepsilon_0}\nabla \times \nabla \times \left[\Delta\varepsilon(r,t) \bullet \mathrm{E}\right] + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\Delta\varepsilon(r,t) \bullet \mathrm{E}\right], \tag{5}$$

where E on the RHS is actually the sum of all fields involved in an interaction.

Consider now the consequences of this new nonlinear wave equation to SBS. According to the physical nature of the phenomenon, as described in the introduction, the only source that contributes to the SBS-generated Stokes field is the reflection of the pump field, $\vec{\mathrm{E}}_p$, by an electrostrictively induced acoustic wave. This means that effects of linear scattering attributed to $\Delta\varepsilon_L(r,t)$, and various nonlinear effects, other than that contributing to SBS, represented through $\vec{\mathrm{P}}_{NL}$ are negligible. To proceed further we will take into consideration the fact that displacement of the medium, which is the variation of the medium's density, $\delta\vec{\rho}(r,t)$, in an acoustic wave is directional, namely it can be along or perpendicular to the propagation direction, known as longitudinal and transverse acoustic waves respectively. In such a case $[\Delta\varepsilon(r,t) \bullet \mathrm{E}]$ on the RHS of Eq. (5) takes the form of the vector multiple, $[\Delta\vec{\varepsilon}(r,t) \times \vec{\mathrm{E}}_p] = (\partial\varepsilon/\partial\rho)[\delta\vec{\rho}(r,t) \times \vec{\mathrm{E}}_p]$, where variation of the material's permittivity is now a vector, $\Delta\vec{\varepsilon}(r,t)$. Consequently, the wave equation for the scattered Stokes field takes the form,

$$\nabla^2\vec{\mathrm{E}}_S - \frac{\varepsilon_0}{c^2}\frac{\partial^2 \vec{\mathrm{E}}_S}{\partial t^2} \cong \frac{1}{\varepsilon_0}\left\{\nabla\left(\nabla\left[\Delta\vec{\varepsilon}(r,t) \times \vec{\mathrm{E}}_p\right]\right) - \nabla^2\left[\Delta\vec{\varepsilon}(r,t) \times \vec{\mathrm{E}}_p\right]\right\} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\Delta\vec{\varepsilon}(r,t) \times \vec{\mathrm{E}}_p\right]. \tag{6}$$

Since in conventional SBS $\Delta\vec{\varepsilon}(r,t)$ is attributed to *longitudinal acoustic waves*, the term $\nabla(\nabla[\Delta\vec{\varepsilon}(r,t) \times \vec{\mathrm{E}}_p])$ is nonzero. The last two terms on the RHS of Eq. (6), which are fully analogous with the propagation terms for the existing $\vec{\mathrm{E}}_S$ on the LHS of this equation, describe the propagation of the newly generated Stokes field, $[\Delta\vec{\varepsilon}(r,t) \times \vec{\mathrm{E}}_p]/\varepsilon_0$. Through introducing a new Stokes field, $\vec{\mathrm{E}}'_S$, on the LHS of Eq. (6) as $\vec{\mathrm{E}}'_S = \vec{\mathrm{E}}_S + [\Delta\vec{\varepsilon}(r,t) \times \vec{\mathrm{E}}_p]/\varepsilon_0$, we get,

$$\nabla^2\vec{\mathrm{E}}'_S - \frac{\varepsilon_0}{c^2}\frac{\partial^2 \vec{\mathrm{E}}'_S}{\partial t^2} \cong \frac{1}{\varepsilon_0}\nabla\left(\nabla\left[\Delta\vec{\varepsilon}(r,t) \times \vec{\mathrm{E}}_p\right]\right). \tag{7}$$

So the RHS of Eq. (7), which no longer has time derivatives, describes the source of the additional Stokes field, $[\Delta\vec{\varepsilon}(r,t) \times \vec{\mathrm{E}}_p]/\varepsilon_0$, as reflection of the pump wave by the grating. This equation is exactly the wave equation for the propagation of a scattered EM wave in an inhomogeneous medium when its dielectric function varies only in space, $\Delta\vec{\varepsilon}(r)$, [5]. When $\Delta\vec{\varepsilon}(r)$ varies periodically in space Eq. (7) is then the basic equation for propagation of an EM wave in media with Bragg gratings. The specific feature of SBS is that the medium's permittivity varies in both space and time since it is a grating moving along the propagation direction of the pump field; it is this movement which results in the Stokes frequency shift of the scattered radiation.

The longitudinal variation of the medium's density, $\delta\vec{\rho}(r,t)$, is governed by the driven acoustic wave equation [6],

$$\frac{\partial^2\delta\vec{\rho}}{\partial t^2} - v_s^2\nabla^2\delta\vec{\rho} - \Gamma\nabla^2\frac{\partial\delta\vec{\rho}}{\partial t} = -\rho_0\frac{\partial\varepsilon}{\partial\rho}\frac{1}{16\pi}\nabla^2\left[\vec{\mathrm{E}}_p(r,t)\vec{\mathrm{E}}'^{*}_S(r,t)\right], \tag{8}$$

where Γ is the acoustic damping parameter and ρ_0 is the equilibrium density of the medium. The multiple of the pump and complex-conjugated Stokes fields on the RHS of Eq. (8) describes the interference pattern, which travels with velocity, v, determined by the ratio $\Omega/|\vec{q}|$, where Ω is the frequency difference of the pump and Stokes waves, $\Omega = \omega_p - \omega_S$, and $\vec{q}$ is the wave-vector of the grating, which is the sum of the pump and Stokes wave-vectors, $\vec{q} = \vec{k}_p + \vec{k}_S$. This pattern, through electrostriction, induces in the medium the moving periodic grating of density variation, $\delta\vec{\rho}(r,t)$. When v coincides with the equilibrium sound wave velocity in the medium, v_s, the amplitude of the induced grating is resonantly enhanced. This resonance is known as a *wave resonance* [7]. The frequency difference, Ω, at which $v = v_s$, is the well known Brillouin frequency, $\Omega_B = |\vec{q}|v_s$. The behavior of the grating is in essence that of a driven damped resonant oscillator, [11], though in the spatial rather than temporal domain; when $\Omega \neq \Omega_B$ the driving force in Eq. (8) still induces the grating, the amplitude of which decreases with increasing difference between v and v_s, but the novelty of the grating is that it now moves with reduced or increased velocity, $v = \Omega/|\vec{q}| \neq v_s$. A noteworthy consequence of this is that the interaction of, for example, monochromatic pump radiation with nonmonochromatic Stokes radiation results in the creation of a set of gratings moving at different velocities, v, which therefore in turn create different spectral components in the Stokes signal due to the Doppler effect. This feature differs substantially from that described in conventional SBS theory where a single fixed sound velocity, v_s, is pre-supposed for all excited gratings and the interaction of a monochromatic pump and nonmonochromatic Stokes radiation results in creation of a band of frequencies