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Ordered and isomorphic mapping of periodic structures in the parametrically forced logistic map

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ABSTRACT

We investigate the periodic domains found in the parametrically forced logistic map, the classical logistic map when its control parameter changes dynamically. Phase diagrams in two-parameter spaces reveal intricate periodic structures composed of patterns of intersecting superstable orbits curves, defining the cell of a periodic window. Cells appear multifoliated and ordered, and they are isomorphically mapped when one changes the map parameters. Also, we identify the characteristics of simplest cell and apply them to other more complex, discussing how the topography on parameter space is affected. By use of the winding number as defined in periodically forced oscillators, we show that the hierarchical organization of the periodic domains is manifested in global and local scales.

1. Introduction

In dynamical systems investigation, parameter space bifurcations have been nowadays crucial to discover and understand new phenomena, in particular in studies of periodic domains organization, either discrete- or continuous-time systems. Early and pioneering studies employing stability diagrams in two-parameter spaces [1–3] have emphasized the genesis and the aligning of periodic structures (shrimps), while more recent works explore new and interesting features such as the periodic windows replication [4], the torsion-adding phenomena and the asymptotic winding numbers [5], both in periodically forced oscillators, and the spiral-like periodicity hub in circuits [6–9] and in the Rössler system [10].

Other studies show that, varying two parameters in continuous-time systems, regularities between chaotic and periodic phases are observed (1) repeating isomorphically as in the Duffing system [11], and (2) exhibiting hierarchical ordering as in sigmoidal maps [12], in mixed-mode oscillation distributions [13–15], in bifurcations of two coupled FitzHugh–Nagumo oscillators [16], in a damped-forced oscillator [17], in a ecological [18] and a cancer models [19], and in the driven Josephson Junction [20]. In this work, we show how a simple one-dimensional discrete-time system, named parametrically forced logistic map, can exhibit simultaneously these two features — the isomorphic repetition and the hierarchical organization — since three parameters of the map are varied.

The parametrically forced logistic map is a variant of the known logistic map [21] when its control parameter is varied dynamically (discretely in time). This map is similar to a dissipative periodically driven oscillator, since its discretely varied control parameter emulates a time-dependent driven force in continuous systems [22]. Markus and Hess [23] and Markus [24] have used the parametrically forced logistic map to show multifoliated structures in two-dimensional parameter spaces. Again in the parameter space, periodic structures and the basin of attraction were obtained by Baptista and Caldas [25] for a modulated version of the forced logistic map, while Kuznetsov and Savin [26] investigated the chaos border of typical structures taking binary sequences as a perturbed signal. Recently, Kumeno et al. [27] have used two coupled parametric forced logistic maps to investigate the synchronization phenomena and the basin of attraction. However, the organization of the existing complex structures in periodic domains to the parametrically forced logistic map remains an open question.

We constructed phase diagrams for the parametrically forced logistic map in two ways: by Lyapunov exponents and winding numbers. Lyapunov's version of phase diagrams yields a complex pattern of superstable orbits curves, unique in each periodic domain. The superstable curves intersect in specific points defining

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the basic element discussed in this work, named the cell of a periodic window. The cell is completely described by the winding number's version of the phase diagrams, the winding number defined by the ratio of two frequencies, in analogy to periodically forced oscillators. The preservation of the winding numbers in isomorphic structures, observed along successive parameter spaces, shows that some cell components are repeatedly remapped in a recurrent way, which we illustrate in some cases. From these recurrent cells, we extract sequences of winding numbers that show how the periodic domains are hierarchically organized. And finally, we demonstrate that these sequences of winding numbers, appearing globally in the periodic domains, are manifested locally in unitary cells as well.

2. Forced logistic map and winding number

The forced logistic map is a variant of the traditional logistic map in which the control parameter varies in a time-dependent manner. The map is defined by

$$x_{i+1} = \lambda_i x_i (1 - x_i), \tag{1}$$

where the parameter λ_i changes dynamically (with i) in any P-length sequence. We allow varying the parameter either two values, $\lambda_i = a$ or b [23,24,26,27], generating sequences

$$\{\lambda_i\} = \{\underbrace{a^{p_1} b^{p_2}}_{P\ \text{terms}} a^{p_1} b^{p_2} \ldots\}, \tag{2}$$

where p_1 and p_2 are integers and $p_1 + p_2 = P$. Eqs. (1) and (2) emulate two characteristics of a continuous-time damped driven oscillator: the "$(1-x)$" term emulates the damping factor, and the sequence $\{\lambda_i\}$ the driven force oscillating either the states 'a' and 'b' under forcing period P.

We define the winding number w_γ, for a trajectory γ, as a ratio of two frequency (as in periodically forced oscillators) [5, 36–41]

$$w_\gamma = \frac{\Omega_\gamma}{f_P}, \tag{3}$$

where Ω_γ is the visiting branch frequency and f_P the "driven force" frequency. The Ω_γ is the (mean) number of times that the points x_i in Eq. (1) visit the branch of first return map (x_{i+1} vs. x_i) with negative slope by N iterations [28], or

$$\Omega_\gamma = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} \theta_i, \tag{4}$$

were $\theta_i = 0$ if $x_i < 1/2$ or $\theta_i = 1$ if $x_i > 1/2$. Observe that the above limit results in $\Omega_\gamma = n/p_\gamma$: the "torsions" n (i.e. number of visits to the negative branch of the first return map) by orbital period p_γ [29–35]. As $f_P = 1/P$, it follows from Eq. (3) that

$$w_\gamma = P\Omega_\gamma. \tag{5}$$

Since $p_\gamma = mP$, m is an integer in a periodically forced system, it results for the visiting branch frequency in $\Omega_\gamma = n/p_\gamma = n/(mP)$. Taking this result in Eq. (5), we have

$$w_\gamma = \frac{n}{m}. \tag{6}$$

Note that Eq. (6) agrees with definition of winding number as a ratio of two integers (m and n) [29–32]. Finally, Eq. (5) is a valuable tool to identify how a periodic structure is interchanged between different periodic domains.

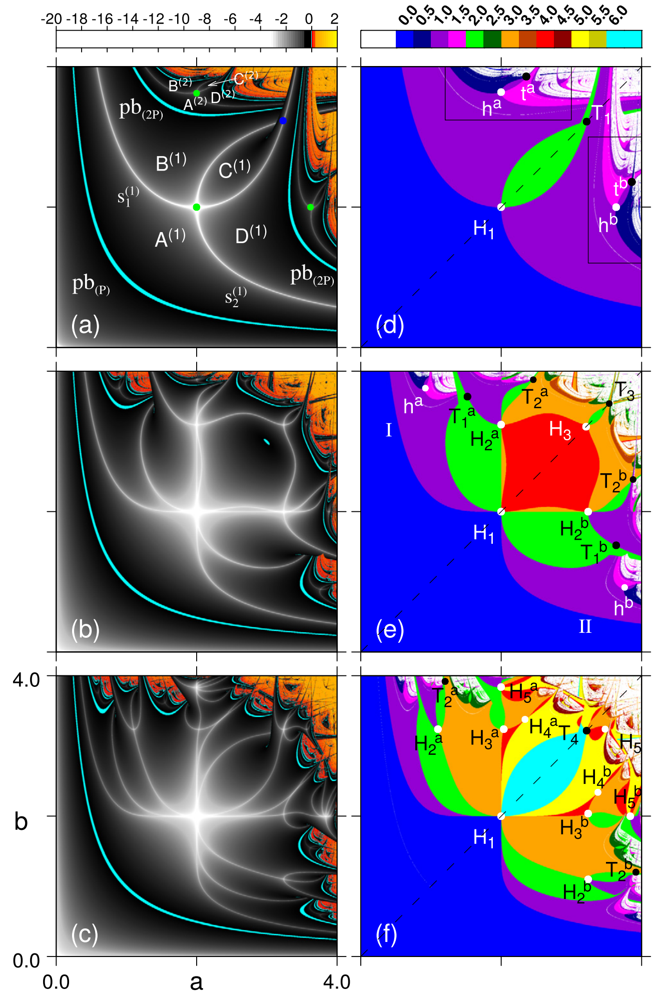

Fig. 1. Lyapunov (scales in bits · s^{-1}) and winding numbers phase diagrams for Eq. (1) for symmetric sequences $\{\lambda_i\}$: $\{a^1b^1\ldots\}$ in (a) and (d); $\{a^2b^2\ldots\}$ in (b) and (e); and $\{a^3b^3\ldots\}$ in (c) and (f). On Lyapunov diagrams (first column) red–orange–yellow colors indicate positive exponents (chaotic behavior); white–grey–black indicate negative exponents (periodic). On winding numbers diagrams (second column) values are depicted by color box, white color representing chaos. In (a) we highlight the *mother-cell* (center) and the *daughter-cell* (top), each cell shows the quadrants around the head point (in green color, see text). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3. Parameter space structure

We allow varying the parameter λ_i in Eq. (1) by any (fixed) periodic sequence of a's and b's, and we analyzed how the asymptotic stable behavior evolves in an (a, b)-parameter space. In fact, by varying λ_i three parameters are varied: the two "driven states" a and b, and the forcing period P. Fig. 1 and Fig. 2 show six charts where the Lyapunov exponents were calculated for the dynamics of Eq. (1). In Fig. 1, first column, we have three diagrams for symmetric sequences $\{\lambda_i\}$ [$p_1 = p_2$, see Eq. (2)]: $\{a^1b^1\ldots\}$ [$P = 2$, Fig. 1(a)]; $\{a^2b^2\ldots\}$ [$P = 4$, Fig. 1(b)] and $\{a^3b^3\ldots\}$ [$P = 6$, Fig. 1(c)]. Fig. 2, first column, shows diagrams for asymmetric sequences $\{\lambda_i\}$ ($p_1 \neq p_2$): $\{a^1b^2\ldots\}$ [$P = 3$, Fig. 2(a)]; $\{a^1b^3\ldots\}$ [$P = 4$, Fig. 2(b)] and $\{a^1b^4\ldots\}$ [$P = 5$, Fig. 2(c)]. Lyapunov diagrams in Figs. 1 and 2 were constructed with a grid of $1200 \times 1200 = 1.44 \times 10^6$ values of a and b. For each pair (a, b) we restart the iteration of Eq. (1) from fixed $x_0 = 0.50$ as the initial value, and calculate the Lyapunov exponent after 2.0×10^3 iterations as a transient. Shaded white–grey–black colors denote periodic behavior that corresponds to the negative exponents, while the posi-