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A generalized Clebsch transformation leading to a first integral of Navier–Stokes equations

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ABSTRACT

In fluid dynamics, the Clebsch transformation allows for the construction of a first integral of the equations of motion leading to a self-adjoint form of the equations. A remarkable feature is the description of the vorticity by means of only two potential fields fulfilling simple transport equations. Despite useful applications in fluid dynamics and other physical disciplines as well, the classical Clebsch transformation has ever been restricted to inviscid flow. In the present paper a novel, generalized Clebsch transformation is developed which also covers the case of incompressible viscous flow. The resulting field equations are discussed briefly and solved for a flow example. Perspectives for a further extension of the method as well as perspectives towards the development of new solution strategies are presented.

1. Introduction

For inviscid flow, Clebsch [1,2] proposed a non-standard potential representation for the velocity field,

$$\vec{u} = \nabla\Phi + \alpha\nabla\beta\,, \tag{1}$$

in terms of the so-called Clebsch variables Φ, α, β. From a mathematical viewpoint, the potential representation (1) is a decomposition of the velocity field into a curl-free part $\nabla\Phi$ and a helicity-free part $\alpha\nabla\beta$. This decomposition is not unique; by applying the gauge transformation:

$$\begin{aligned} \Phi &\longrightarrow \Phi' = \Phi + f(\alpha,\beta,t) \\ \alpha &\longrightarrow \alpha' = g(\alpha,\beta,t) \\ \beta &\longrightarrow \beta' = h(\alpha,\beta,t) \end{aligned} \tag{2}$$

an equivalent set of Clebsch variables Φ', α', β' is given if and only if the functions f, g, h fulfill the two PDE [3]:

$$\frac{\partial f}{\partial\beta} + g\frac{\partial h}{\partial\beta} = \alpha\,, \tag{3}$$

$$\frac{\partial f}{\partial\alpha} + g\frac{\partial h}{\partial\alpha} = 0\,. \tag{4}$$

The benefit of the Clebsch transformation becomes apparent by application to Euler's equations for inviscid flows:

$$\begin{aligned} \vec{0} &= \frac{\mathrm{D}\vec{u}}{\mathrm{D}t} + \nabla[P+U] \\ &= \nabla\left[\frac{\partial\Phi}{\partial t} + \alpha\frac{\partial\beta}{\partial t} + \frac{\vec{u}^2}{2} + P + U\right] + \frac{\mathrm{D}\alpha}{\mathrm{D}t}\nabla\beta - \frac{\mathrm{D}\beta}{\mathrm{D}t}\nabla\alpha\,. \end{aligned} \tag{5}$$

By $P = \int \varrho^{-1}\mathrm{d}p$ the pressure function is denoted and by U the potential energy of the external force. The operator $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \vec{u}\cdot\nabla$ is the material time derivative. Being basically of the form

$$\nabla[\cdots] + [\cdots]\nabla\alpha + [\cdots]\nabla\beta = \vec{0}\,, \tag{6}$$

this vector equation can be decomposed according to

$$\frac{\partial\Phi}{\partial t} + \alpha\frac{\partial\beta}{\partial t} + \frac{\vec{u}^2}{2} + P + U = F(\alpha,\beta,t) \tag{7}$$

$$\frac{\mathrm{D}\alpha}{\mathrm{D}t} = -\frac{\partial F}{\partial\beta} \tag{8}$$

$$\frac{\mathrm{D}\beta}{\mathrm{D}t} = \frac{\partial F}{\partial\alpha} \tag{9}$$

with an unknown function $F(\alpha,\beta,t)$. By making use of the gauge transformation (2), however, $F \to 0$ can be reached. The above three scalar field equations are a first integral of Euler's equations and self-adjoint, their most intriguing feature, however, is that the vorticity

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$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{u} = \frac{1}{2}\nabla\alpha \times \nabla\beta \tag{10}$$

is given by the two scalar fields α, β, only. Hence, the vortex dynamics is reduced to the two transport equations (8), (9).

It should be mentioned that for an arbitrary velocity field $\vec{u}$ the existence of the Clebsch variables Φ, α, β is surely given only locally. Their global existence depends on the topological features of the flow: in case of a non-vanishing integral of helicity, for example for flows with closed vortex lines that form linked rings or with isolated points of zero vorticity global existence is not given. For details we refer e.g. to [4,5]. In case of global non-existence, completeness of the Clebsch representation may be reached by additional pairs of variables, like: $\vec{u} = \nabla\Phi + \alpha_1\nabla\beta_1 + \alpha_2\nabla\beta_2 + \cdots$. Subsequently attention is paid only to the classical form (1).

The Clebsch transformation has also been applied to different physical problems, for instance to baroclinic flow [6], Maxwell equations in classical electrodynamics [7], to Magnetohydrodynamics [8] and even quantum theory within the context of a quantization of vortex tubes [9]. Viscous flow, however, has not yet been formulated in terms of Clebsch variables to our best knowledge. In Sect. 2 this problem is analysed and a generalization of the method is proposed which successfully applies to viscous flow, leading to a first integral of Navier–Stokes equations. In Sect. 3 the solving procedure is demonstrated by analytical means for a stagnation flow. Further perspectives for application of the method, e.g. to dislocations in solid mechanics, are briefly discussed in Sect. 4.

2. A generalized Clebsch transformation

2.1. Non-applicability of classical Clebsch transformation to viscous flow

We consider the Navier–Stokes equations with continuity equation,

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} - \nu\Delta\vec{u} + \nabla\left[\frac{p}{\varrho} + U\right] = \vec{0}\,, \tag{11}$$

$$\nabla\cdot\vec{u} = 0\,, \tag{12}$$

assuming incompressible flow according to (12), due to the classical theory of viscous flow [2]. We remark that in the more general case of compressible flow the continuity equation reads $\varrho\nabla\cdot\vec{u} = -\mathrm{D}\varrho/\mathrm{D}t$ and the Eqns. (11) have to be replaced likewise by their more general form, frequently called *Navier–Stokes–Duhem equations*, see e.g. [10].

The essential problem inhibiting the application of the Clebsch transformation on viscous flow is due to the friction force density $-\nu\Delta\vec{u}$ in the Navier–Stokes equations. Written in terms of the Clebsch variables, it reads:

$$\begin{aligned}-\nu\Delta\vec{u} &= \nu\Delta\beta\nabla\alpha - \nu\Delta\alpha\nabla\beta \\ &\quad - \nu\left(\nabla\alpha\cdot\nabla\right)\nabla\beta + \nu\left(\nabla\beta\cdot\nabla\right)\nabla\alpha\,.\end{aligned} \tag{13}$$

Obviously, only two of the above four terms fit into the scheme (6), whereas the other two, subsumed to a vector field

$$\vec{a} := \nu\left(\nabla\beta\cdot\nabla\right)\nabla\alpha - \nu\left(\nabla\alpha\cdot\nabla\right)\nabla\beta\,, \tag{14}$$

are of a mathematical form incompatible with (6). More general, the problem of finding a decomposition of the form (6) with prescribed Clebsch variables α, β is handled for an *arbitrary* vector field $\vec{a}$ in the following.

2.2. Solution procedure

We first introduce an auxiliary field ξ, fulfilling the first order PDE

$$\vec{\omega}\cdot\nabla\xi = \vec{\omega}\cdot\vec{a}\,, \tag{15}$$

with vorticity ω given according to (10). This implies the identity

$$\begin{aligned}\vec{\omega}\times\left(\vec{\omega}\times\left[\vec{a}-\nabla\xi\right]\right) &= \vec{\omega}\left(\vec{\omega}\cdot\left[\vec{a}-\nabla\xi\right]\right) - \left[\vec{a}-\nabla\xi\right]\vec{\omega}^2 \\ &= -\vec{\omega}^2\left[\vec{a}-\nabla\xi\right]\end{aligned} \tag{16}$$

and therefore the decomposition of the difference $\vec{a}-\nabla\xi$ as

$$\begin{aligned}\vec{a}-\nabla\xi &= \frac{\left(\vec{\omega}\times\left[\vec{a}-\nabla\xi\right]\right)\times\vec{\omega}}{\vec{\omega}^2} \\ &= \frac{\left(\vec{\omega}\times\left[\vec{a}-\nabla\xi\right]\right)\times\left(\nabla\alpha\times\nabla\beta\right)}{2\vec{\omega}^2} \\ &= \frac{\left(\vec{\omega}\times\left[\vec{a}-\nabla\xi\right]\right)\cdot\nabla\beta}{2\vec{\omega}^2}\nabla\alpha - \frac{\left(\vec{\omega}\times\left[\vec{a}-\nabla\xi\right]\right)\cdot\nabla\alpha}{2\vec{\omega}^2}\nabla\beta\end{aligned} \tag{17}$$

i.e. as linear combination of $\nabla\alpha$ and $\nabla\beta$. Obviously, the decomposition (17) can be applied to an arbitrary vector field $\vec{a}$ in order to reach the form (6).

Like the Clebsch variables Φ, α, β, the auxiliary field ξ is not uniquely given, since any particular solution ξ_{p} of the inhomogeneous linear first order PDE (15) can be superposed with any solution ξ_{h} of the respective homogeneous PDE $\vec{\omega}\cdot\nabla\xi_{\mathrm{h}} = 0$. Since three independent solutions are given by α, β and t, the mathematical theory of linear first order PDE implies $\xi_{\mathrm{h}} = F(\alpha, \beta, t)$ for an arbitrary function F. As a consequence,

$$\xi \longrightarrow \xi' = \xi + F(\alpha, \beta, t) \tag{18}$$

is a gauge transformation for the auxiliary field which is used subsequently for a favourable form of the resulting equations.

2.3. First integral of Navier–Stokes equations

The Navier–Stokes equations (11) contain identical mathematical terms as Euler's equations (5), apart from the pressure function taking the special form $P = p/\varrho$ for incompressible flow, plus the friction term $-\nu\Delta\vec{u}$. Therefore, the Clebsch transformation delivers the three equations (7)–(9), supplemented by the terms resulting from the decomposition of the friction term according to (13) and (15), (17). Using (18), the function F appearing in (7)–(9) is set to zero by gauging, leading finally to the set of the three scalar field equations

$$\frac{\partial\Phi}{\partial t} + \alpha\frac{\partial\beta}{\partial t} + \frac{\vec{u}^2}{2} + \frac{p}{\varrho} + U + \xi = 0\,, \tag{19}$$

$$\frac{\mathrm{D}\alpha}{\mathrm{D}t} - \nu\Delta\alpha - \frac{\vec{\omega}\times\left[\vec{a}-\nabla\xi\right]}{2\vec{\omega}^2}\cdot\nabla\alpha = 0\,, \tag{20}$$

$$\frac{\mathrm{D}\beta}{\mathrm{D}t} - \nu\Delta\beta - \frac{\vec{\omega}\times\left[\vec{a}-\nabla\xi\right]}{2\vec{\omega}^2}\cdot\nabla\beta = 0\,, \tag{21}$$

where again $\vec{\omega}$ and $\vec{a}$ have been used as abbreviations according to (10), (14). Thus, a first integral of Navier–Stokes equations has been constructed, based on the generalized Clebsch transformation: Eq. (19) is a generalization of Bernoulli's equation, whereas the two evolution equations (20), (21) for the vortex potentials α, β reveal the generic type of convection–diffusion equations with additional nonlinear coupling terms. The set of equations is completed by the PDE (15) for the auxiliary field ξ and the continuity equation (12). The latter one reads in terms of Clebsch variables [11]: $\Delta\Phi + \alpha\Delta\beta + 2\nabla\alpha\cdot\nabla\beta = 0$.