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Can recurrence networks show small-world property?

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ABSTRACT

Recurrence networks are complex networks, constructed from time series data, having several practical applications. Though their properties when constructed with the threshold value ϵ chosen at or just above the percolation threshold of the network are quite well understood, what happens as the threshold increases beyond the usual operational window is still not clear from a complex network perspective. The present Letter is focused mainly on the network properties at intermediate-to-large values of the recurrence threshold, for which no systematic study has been performed so far. We argue, with numerical support, that recurrence networks constructed from chaotic attractors with ϵ equal to the usual recurrence threshold or slightly above cannot, in general, show small-world property. However, if the threshold is further increased, the recurrence network topology initially changes to a *small-world* structure and finally to that of a classical random graph as the threshold approaches the size of the strange attractor.

1. Introduction

In the last two decades, complex network theory has emerged as a popular tool for analyzing complex and spatially extended systems [1–3]. It has found applications in a wide range of fields including sociology [4,5], communication [6,7] and biological sciences [8,9]. The theory of complex networks initially started with random graphs (RG) studied in detail by Erdős and Rényi (E–R) [10]. For RGs, there is a fixed probability p for two nodes being connected and one can show that for a sufficiently large number of nodes N, the degree distribution $P(k)$ tends to a Poissonian.

The E–R model guided our thinking about complex networks for many decades until the discovery by Barabasi and co-workers [11,12] nearly two decades back that the topology and structure of most networks around us are radically different. For example, many networks in the real world such as, the World Wide Web (WWW) [13], networks of social interactions [14], protein and metabolic networks [15] and technological networks [16], tend to obey certain self-organizing principles in their evolution either due to some inherent property of the system or due to the nature of human interactions as in social networks. The topology of such networks shows a *scale invariance* with the degree distribution obeying a power law, $P(k) \propto k^{-\gamma}$, and the value of γ is found to be typically between 2 and 3. The discovery of such *scale-free* (SF) networks triggered a lot of interest in the theory of complex networks.

Along with the discovery of SF networks, the concept of *small-world* (SW) networks was introduced by Watts and Strogatz [17]. Though the classical RGs are amenable to a great deal of mathematical analysis, they are poor models as far as real networks are concerned. Firstly, they show poor clustering and their clustering coefficient (CC) is directly proportional to p. Secondly, for a given p, as the number of nodes N increases, the average degree $< k >$ also increases correspondingly. Consequently, for p above a threshold value, the characteristic path length (CPL) of RGs remains very small and independent of N.

Watts and Strogatz (W–S) showed that, starting from a ring lattice of N nodes with nearest neighbor coupling and randomly re-wiring a small fraction β of edges, results in a complex network with high CC compared to the RG and small CPL comparable to that of a RG for a range of values of β. Moreover, for such networks, as N increases, the CPL increases only as $\log N$. Thus the W–S model displays many characteristic properties of real world networks and provides one possibility of obtaining the SW property, often found in real world networks, with different levels of complexity by tuning β.

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The above developments resulted in complex networks and the related measures being applied as tools in many areas of research. The most recent among them is the analysis of time series from dynamical systems using statistical measures of complex networks. To study many dynamical processes in the real world, one often resorts to the analysis of time series obtained from the system. An area of special interest is where the underlying system appears to show deterministic nonlinear behavior. The methods and measures of nonlinear time series analysis and chaos theory [18,19] are commonly used in such cases. In the last few years, measures based on complex networks have gained a lot of importance in nonlinear time series analysis. All such measures propose a mapping from the time series domain to the network domain and then proceed to characterize the dynamical system in terms of the statistical measures of the resulting complex network. By doing this, one expects to resolve complimentary features that are not captured by conventional methods of time series analysis, especially the structural and topological properties of the underlying chaotic attractor.

Even though several methods have been proposed [20–22] to convert time series into networks, an approach incorporating the generic property of *recurrence* [23] of a dynamical system has been prominently applied for the conversion of time series into networks. In this method, the time series is first embedded in a suitable dimension M using time delay co-ordinates [24] to reconstruct the attractor. Every point on the attractor is identified as a node and the network can be constructed in two ways, either by taking a fixed number of nearest neighbors [25] or by taking a fixed hyper-sphere of radius ϵ with the point as the center. In this work, we consider the second method for the construction of the network where a reference node $\imath$ is connected to another node $\jmath$ if the Euclidean distance d_{ij} between the corresponding points on the attractor in the reconstructed space is less than or equal to the recurrence threshold ϵ, that is, if $d_{ij} \leq \epsilon$. The resulting complex network, called the ϵ – recurrence network or simply recurrence network (RN) [26,27], has been shown to have great potential for a wide range of practical applications, from identifying critical transitions in dynamical systems [28] to the classification of cardio-vascular time series [29]. Note that, by construction, the RN is an undirected and unweighted graph with a symmetric and binary adjacency matrix $\mathcal{A}$, with elements $A_{ij} = 1$ or 0, depending on whether the two nodes $\imath$ and $\jmath$ are connected or not.

Though RNs and related statistical measures are widely applied in nonlinear time series analysis, their properties, as the threshold is increased, are not fully understood from a complex network perspective. It is well known that all RNs have two properties in common. Firstly, the degree distribution of every RN is unique and is closely related to the probability density variations over the embedded attractor from which it is mapped [30]. We will discuss this in detail below. Secondly, there is an absence of long range connections as the maximum edge length is limited by the recurrence threshold ϵ. By definition, RNs are random geometric graphs (RGG) in the considered system's phase space [30–32]. Here RGGs are RGs where each vertex is randomly assigned co-ordinates in some metric space according to some prescribed probability distribution function, and vertices are connected if and only if they are separated by less than a certain maximum distance [33].

In this Letter, we numerically investigate the specific properties of RNs using three primary measures of a complex network, the degree distribution, the CC and the CPL. In particular, we consider the range of threshold values beyond the small operational window usually used for the construction of the RN and check whether the resulting network can show the properties of either RGs or SF networks or behave as a small-world network.

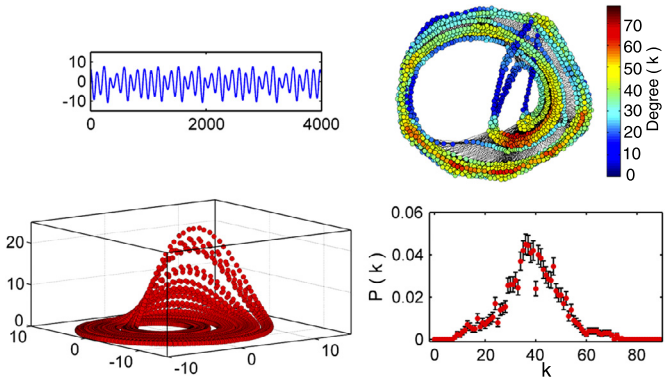

Fig. 1. Construction of the RN from the time series of the standard Rössler attractor's y – component using time step $\Delta t = 0.05$ with time delay $\tau = 24$ and $M = 3$. The time series and the embedded attractor are shown (top and bottom, respectively) in the left panel while the RN and its degree distribution are shown in the right panel. The RN is constructed taking every point on the attractor as a node and connecting every node to all other nodes within a recurrence threshold of $\epsilon = 0.1$ (see text). Error bars of $P(k)$ originate from the fact that for a network with N nodes, the number n of nodes with a given degree k has a standard error of $\sqrt{n(k)}$. For $n(k) \to 0$, its value is normalized to 1, the minimum count.

2. Numerical results

All numerical simulations are done using $N = 2000$ nodes, unless otherwise mentioned. For the construction of SF networks, we use the specific scheme of preferential attachment discussed in detail in [34]. Here we start with a small number of nodes m_0. A new node is then added which gets connected to m number of existing nodes. This process is repeated, increasing the number of edges by m for each newly connected node. By changing either m_0 or m we can construct SF networks with different γ. Here we do both and construct the SF networks using 3 values for m_0 (2, 4 and 10) and in each case, use 3 different values for m, namely, 1, 2 and 4. Moreover, the RGs are constructed for different values of p. Time series from several standard low-dimensional chaotic systems are used for the construction of RNs. For continuous systems (flows) in 3D, we use the embedding dimension $M = 3$ and for discrete systems (maps) in 2D, we use $M = 2$. For all systems, we use the time series from the y – component and for all continuous systems, the time step Δt used for numerical integration is 0.05 with the time delay computed by the automated algorithmic scheme proposed by us [35]. The values of time delay used for the Rössler, Lorenz, Duffing and Ueda systems are 24, 6, 25 and 13 respectively.

To get a quantitatively comparable value for the percolation threshold ϵ and to make the comparison between systems possible, we first transform the time series to a uniform deviate so that the size of the reconstructed attractor is re-scaled into a unit cube in M dimensions. This transformation is not a trivial rescaling as it stretches the embedded attractor uniformly in all directions. We have already shown [35,36] how effective this transformation is in computing conventional measures such as, correlation dimension D_2 and entropy K_2, for low as well as high-dimensional systems from time series [37]. As a result of the uniform deviate transformation, we have found that it is possible to have a small identical range of recurrence threshold $\Delta\epsilon$, that can be taken as the operational window for constructing the RN from time series of different systems for a given embedding dimension M [38]. Here we choose the value of ϵ as the minimum of this range where the giant component of the RN just appears, as suggested by Donges et al. [32]. The value of ϵ is found to be 0.06 for $M = 2$ and 0.1 for $M = 3$ for $N \leq 10000$ as discussed in detail in [38]. The RN from a random (white noise) time series is also constructed using $M = 3$ (for comparison with the RNs from chaotic systems) whose degree distribution is Poissonian with $< k > \approx 7$ for the selected ϵ. We find