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On singular and sincerely singular compact patterns

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A B S T R A C T

A third order dispersive equation $u_t + (u^m)_x + \frac{1}{b}[u^a \nabla^2 u^b]_x = 0$ is used to explore two very different classes of compact patterns. In the first, the prevailing singularity at the edge induces traveling compactons, solitary waves with a compact support. In the second, the singularity induced at the perimeter of the initial excitation, entraps the dynamics within the domain's interior (nonetheless, certain very singular excitations may escape it). Here, overlapping compactons undergo interaction which may result in an interchange of their positions, or form other structures, all confined within their initial support. *We conjecture*, and affirm it empirically, that whenever the system admits more than one type of compactons, only the least singular compactons may be evolutionary. The entrapment due to singularities is also unfolded and confirmed numerically in a class of diffusive equations $u_t = u^k \nabla^2 u^n$ with $k > 1$ and $n > 0$ with excitations entrapped within their initial support observed to converge toward a space–time separable structure. A similar effect is also found in a class of nonlinear Klein–Gordon Equations.

1. Introduction

The solitary patterns in a continuous medium, as predicted by the equations of mathematical physics, think solitons, are fairly localized yet their analyticity causes their tails to extend indefinitely. Over two decades ago we have shown [1–3] that truly compact dispersive structures are possible but in order to achieve a true localization, one needs a singularity inducing mechanism which breaks their analytical spell. In this letter we dive deeper into the relations between the nature of such singularities and the resulting compact patterns (few of the aspects of singularity shaped patterns in $1-D$ were briefly touched upon in [3]). Lacking a rigorous proof, the formal analysis is backed by extensive numerical studies to assure that conjectures which stand up to reason, stand to reality as well. Our presentation will be mostly confined to a class of third order partial differential dispersive equations

$$\mathcal{C}_N(m, a+b): \ u_t + (u^m)_x + \frac{1}{b}[u^a \nabla^2 u^b]_x = 0, \quad 1 < n \doteq a+b \tag{1.1}$$

with the subindex N indicating a $N-D$ spatial span. The $1-D$ problem will be discussed in this and the following section, whereas the $N-D$ problem ($N > 1$) will be addressed in sec. 3. Two types of singularities are found with the stronger one, referred henceforth as a *sincere singularity*, being capable of entrapping the dynamics within the initial domain. To stress the ubiquitous aspect of our results we discuss in Appendix B a class of dissipative equations

$$u_t = u^k \nabla^2 u^n, \quad n > 0, k > 1,$$

and find patterns shaped by singularities with entrapment features very similar to ones found in dispersive medium. In particular, apart of the well documented role of singularities to induce compact diffusive patterns which converge to a universal self-similar state [4], we also unfold a sincere singularity which confines the diffusion within its initial support where it approaches very quickly a time–space separable structure. Yet another application, briefly touched upon in the last section, is afforded by a class of nonlinear Klein–Gordon equations.

Returning to Eq. (1.1) [3], we note that in $1-D$ it generalizes the more familiar $\mathcal{K}(m,n)$ equation

$$\mathcal{K}(m,n): \ u_t + (u^m)_x \pm (u^n)_{xxx} = 0, \quad -1 < n, \tag{1.2}$$

and provides a more natural framework to study the scope of nonlinear dispersion. In fact, the $a = b = 1$ case was derived in [5] in the study of suspensions whereas the $m = 3, 4$, with $a = 3$, $b = 1$, cases correspond to a Lagrange map of the K-dV and modified K-dV equations, respectively, see Appendix A for more details. In both (1.1) and (1.2) the various coefficients may, modulus the sign, be normalized at will. In what follows we shall use the following parameters

$$\omega \mathrel{\hat=} 1 + b - a \quad \text{and} \quad \omega_m \mathrel{\hat=} m + b - a. \tag{1.3}$$

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ω plays a crucial role. For instance, if $n=2$ then Eq. (1.1) may be rewritten as

$$u_t + (u^m)_x + uu_{xxx} + \omega u_x u_{xx} = 0.$$

As we shall see shortly, since at $u=0$ the highest order spatial operator degenerates, the lower order terms acquire a crucial impact on the dynamics with $\omega=0$ dividing between two very different regimes. $\omega u_x u_{xx}$ may be viewed as a diffusion with ωu_x playing the role of a diffusion coefficient which changes its sign on the two sides of a pulse. Clearly, a change in the sign of ω dramatically changes the impact of such a term. For $n \neq 2$, and the dispersive part casted as $\left[u^{n-2}(uu_{xx} + D(\omega)u_x^2)\right]_x$, $D(\omega) = (\omega+n-3)/2$, things are essentially the same: ω controls the lower order part of the dispersion.

In addition, when $w \neq 0, 1$, $\int u^\omega$ is conserved, which enables an alternative form of Eq. (1.1)

$$v_t + A(v^{m_*})_x + B\left[v^{a_*}\left(v^{b_*}\right)_{xx}\right]_x = 0 \quad \text{where } v = u^\omega, \tag{1.4}$$

and

$$m_* = \frac{\omega_m}{\omega}, a_* = \frac{3b-a}{2\omega}, b_* = \frac{n}{2\omega}, A = \frac{\omega m}{m+b-a} \text{ and } B = \frac{2\omega}{n}.$$

In the $b=a+1$ case, equation (1.1) admits an Hamiltonian, [3]

$$\mathcal{H} = \int \left[\frac{1}{m+1}u^{m+1} - \frac{1}{2(a+1)^2}\left[\left(u^{a+1}\right)_x\right]^2\right]dx.$$

In addition, if $m=2a+3$ it also conserves

$$\int \left\{xu^2 - 2(2a+3)t\left(\frac{1}{2a+4}u^{2a+4} - \frac{1}{2(a+1)^2}\left[\left(u^{a+1}\right)_x\right]^2\right)\right\}dx,$$

which through the Hamiltonian formulation is related to its scaling symmetry.

Our main conjecture concerning equation (1.1) is: Wherein more than one profile of either traveling or stationary compactons is admissible, only compactons of lesser singularity may be evolutionary whereas the other compactons decompose. Based on formal analysis of the traveling and stationary waves aided by our numerical study we find:

1. $1<\omega$ (i.e., $a<b$): Only the traveling compactons are admissible. No other compact structures were detected.
2. $0<\omega\le 1$ (i.e., $b \le a < b+1$): Both traveling and stationary compactons are in principle admissible. As conjectured, numerical studies reveal that the traveling compactons being of lesser singularity, are the ones to materialize.
3. $\omega<0$ (i.e., $b+1<a$): Both traveling and stationary compactons are in principle admissible. As conjectured, numerical studies confirm that the stationary compactons which in this parametric regime are less singular, are the ones to materialize. Moreover, both the results of [3] (see also Appendix A) and numerical simulations clearly indicate that interaction between overlapping stationary compactons typically results in an interchange of their positions, all confined within their initial support.
4. $\omega=0$ (i.e., $b+1=a$): This is an exceptional case. Though stationary compactons are less singular than the traveling ones which decompose at once, after a while they form a precursor of negative amplitude and blow up. All in all, numerical studies indicate that *in the $\omega=0$ case there are no viable compact structures*.

Summarizing our results: We have verified numerically our conjecture that whenever more than one type of compactons is admissible, the one of lesser singularity may be evolutionary, whereas the other type decomposes. Our simulations of equation (1.1) have been carried using a Local Discontinuous Galerkin method (LDG) [6] and a choice of numerical fluxes based on [13]. To recheck the validity of our results, part of our simulations were redone using a code based on a pseudo-spectral method developed to address nonlinear quintic PDEs [14].

Though not directly related to our interests, we comment on recent inquiries about the well posedness of the initial value problem of the $K(2,2)$ and related equations for initial data which are not uniformly bounded away from zero [11]. Numerical studies reported in [12] present a sequence of narrowing initial pulses which induce a flow that may lose in the H^2 norm its continuous dependence on the initial datum. Though this may evoke challenging questions, we have found that in the realm of our interest, compactly supported initial data which are typically much wider than a single compacton, the issues raised in [12] do not arise. Moreover, not only we have found, using two very different numerical methods, our equations to be well behaved, but in hundreds of numerical simulations both the residuum left after the emergence of compactons out of initial datum, and the residuum left after interaction of compactons, have never shown behavior which could cast any doubt in the validity of our results or of the underlying equations. To farther enhance the credibility of our assertions, we have repeatedly employed in our simulations spatial refinements to eliminate potential spurious phenomena.

2. Compact structures on a line

2.1. Stationary patterns

Setting $u_t=0$ and integrating Eq. (1.1) yields

$$u^m + \frac{1}{b}u^a(u^b)_{xx} = E_0.$$

Multiplying by $u^{-a}(u^b)_x$, integrating again and rearranging yields

$$\frac{1}{\omega_m}u^m + \frac{1}{2}u^{n-2}u_x^2 = \frac{1}{b-a}E_0 + E_1 u^{a-b}, \tag{2.1}$$

where E_0 and E_1 are constants of integration. Eq. (2.1) describes a *2-parameter class of stationary periodic solutions.* For $b>a>0$, to generate candidates for compactification and to avoid an overly singular solution at $u=0$, we set $E_1=0$ and $E_0>0$. To this end we locate the troughs of the periodic wave at $u=0$ where the degeneracy of the highest order operator and the consequent loss of uniqueness enable to form a compact solution by gluing one period between two subsequent troughs with the trivial ground state. Let one of such troughs be located at, say $x=0$, then from Eq. (2.1),

$$u \sim x^{2/n}.$$

A compactification at this point would mean that $u \sim H(x)x^{2/n}$, where $H(x)$ is the Heaviside function. Since, however,

$$u^a(u^b)_{xx} \sim H(x) \;\Rightarrow\; [u^a(u^b)_{xx}]_x \sim \delta(x)$$

it cannot be balanced by the convective counterpart $(u^m)_x$ in (1.1). Thus when $b>a$ no stationary compact solutions seem possible.

If $m>a-b>0$, the least singular solutions call for $E_0=0$ in (2.1) and near a trough of the periodic solution

$$u \sim x^{1/b}. \tag{2.2}$$

To compactify this solution we remove everything but one period between two consecutive troughs. Near the trough $u \sim H(x)x^{1/b}$ and $u^b \sim (x + a_1x^2 + \ldots)H(x)$. Thus