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Homopolar oscillating-disc dynamo driven by parametric resonance

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ABSTRACT

We use a simple model of Bullard-type disc dynamo, in which the disc rotation rate is subject to harmonic oscillations, to analyze the generation of magnetic field by the parametric resonance mechanism. The problem is governed by a damped Mathieu equation. The Floquet exponents, which define the magnetic field growth rates, are calculated depending on the amplitude and frequency of the oscillations. Firstly, we show that the dynamo can be excited at significantly subcritical disc rotation rate when the latter is subject to harmonic oscillations with a certain frequency. Secondly, at supercritical mean rotation rates, the dynamo can also be suppressed but only in narrow frequency bands and at sufficiently large oscillation amplitudes.

1. Introduction

In dynamo experiments, a high driving power is necessary to achieve the self-excitation of the magnetic field. The ensuing liquid metal flow is usually strongly turbulent. In both Riga and Karlsruhe dynamo experiments [1,2], the turbulent fluctuations were partly inhibited by the internal walls, whereas in the Cadarache experiment [3], the absence of such walls resulted in large-scale flow fluctuations [4]. The effect of flow fluctuations on the dynamo threshold has been addressed in several recent studies [5–14]. Solving the kinematic dynamo problem for a given non-stationary flow usually governed by the Navier–Stokes equations shows that turbulence generally has an adverse effect on the dynamo excitation, unless the fluctuations are strong enough to drive the dynamo by themselves without any mean flow. In the latter case, we speak of a fluctuation dynamo [15,16], whose experimental implementation seems hardly feasible because of the high excitation threshold. However the possibility that fluctuations excite the magnetic field by the parametric resonance mechanism [17] cannot be excluded. Parametric resonance has been proposed in the somewhat different context of spiral galaxies as a promoter of bisymmetric magnetic field structure [18–21].

In this Letter, we use a simple model of the Bullard-type disc dynamo [22] to show that the magnetic field can indeed be excited by the parametric resonance mechanism, even when relatively small harmonic oscillations are added to significantly subcritical disc rotation rates.

2. Problem formulation

Consider a Bullard-type disc dynamo [22] which consists of a solid conducting disc rotating with a generally time-dependent angular velocity $z(t)$ about its axis, and a wire twisted around the axle and connected by sliding contacts to the rim of the disc and the axle as shown in Fig. 1. The disc is assumed to be segmented so that azimuthal current can flow only at its rim. This corresponds to the modification of the Bullard disc dynamo suggested by Moffatt in order to eliminate exponential growth of the magnetic field in the limit of a perfectly conducting disc [23]. The system is de-

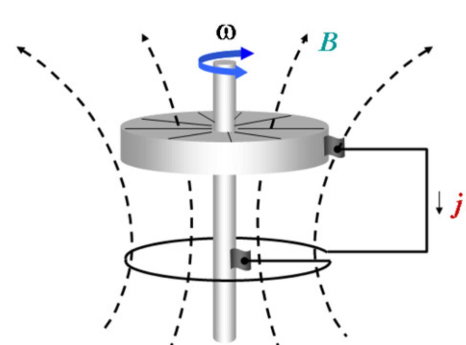

Fig. 1. Sketch of a homopolar disc dynamo.

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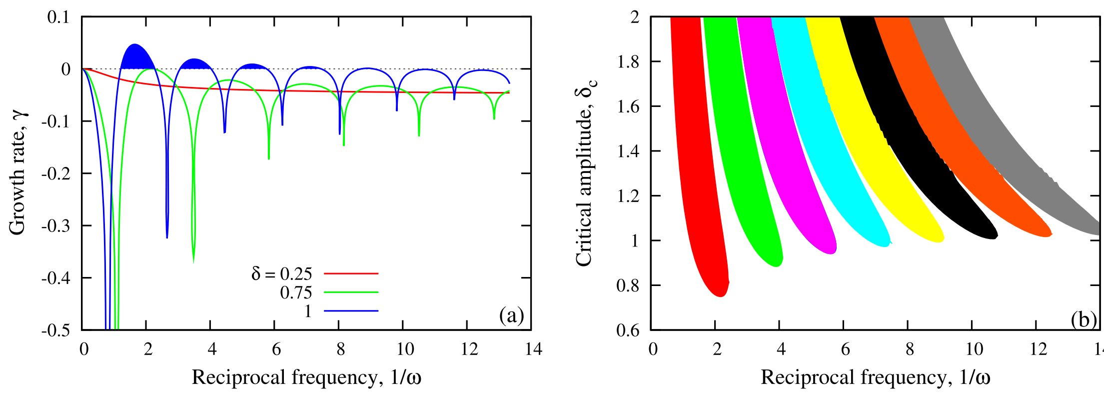

Fig. 2. (a) Growth rate γ versus the reciprocal frequency and (b) the critical amplitude δ_c versus the frequency for the marginal mean rotation rate at $\alpha_0 = 0$.

scribed by the following set of dimensionless equations (for details see Ref. [25]):

$$\dot{x} = r(y - x),$$
$$\dot{y} = xz + mx - (m+1)y,$$
$$\dot{z} = g\big[1 + x\big(mx - (m+1)y\big)\big] - kz, \quad (1)$$

where x and y are magnetic fluxes through the loop made by the wire and the rim of disc, respectively; z is the dimensionless angular velocity of the disc; r accounts for the resistance of the disc relative to that of the loop, and m characterizes the relative mutual inductance of the disc and the loop; the dot stands for the time-derivative d/dt. The disc is driven by a generally time-dependent torque g, and braked by a viscous-type friction characterized by the coefficient k, which is necessary for the structural stability of the system [24]. Henceforth, we assume the friction to be strong with respect to the inertia of the disc accounted for by $\dot{z}$ in Eq. (1), which, thus, results in $z = z_0[1 + x(mx - (m+1)y)]$, where $z_0 = g/k$. The remaining two 1st-order ODEs in (1) can be combined into a single 2nd-order Duffing-type equation [7] with a non-linear friction

$$\ddot{x} + \big(1 + \beta x^2\big)\dot{x} - \alpha x + \lambda x^3 = 0, \quad (2)$$

where x and t are rescaled by $(m+1+r)$ and $(m+1+r)^{-1}$, respectively, and $\alpha = r(z_0 - 1)/(m+1+r)^2$, $\beta = z_0(m+1)(m+1+r)$, and $\lambda = rz_0$. Further, we focus on the evolution of small initial perturbations of the magnetic field characterized by $x \ll 1$, for which Eq. (2) can be linearized by setting $\beta = \lambda = 0$. Then the only remaining parameter α depends directly on the deviation of the disc rotation rate from its critical value $\alpha = 0$. For $\alpha > 0$, a small initial magnetic field starts to grow exponentially provided that the disc rotates steadily [22]. Incidentally we remark that the original one-dimensional disc dynamo model of Bullard [22] corresponds taking $1/r = 0$ (and $k = 0$) in (1), leading to an equation of only 1st-order derivative in x.

In this study, we are interested in how the generation of the magnetic field is affected by the unsteadiness of the disc rotation

$$\alpha = \alpha_0 + \delta\cos(\omega t), \quad (3)$$

which besides the mean part α_0 contains also an oscillatory component with the amplitude δ and the circular frequency ω. Then the linearized Eq. (2) reduces to a damped Mathieu equation

$$\ddot{x} + \dot{x} - \big(\alpha_0 + \delta\cos(\omega t)\big)x = 0.$$

Using the substitution $x(t) = \exp(-t/2)\chi(\omega t/2)$, the equation above can be transformed into the canonical Mathieu equation

$$\ddot{\chi} + \big[a - 2q\cos(2\tau)\big]\chi = 0, \quad (4)$$

where $a = -(1 + 4\alpha_0)/\omega^2$, $q = 2\delta/\omega^2$ and $\tau = \omega t/2$. According to Floquet theory, a particular solution to Eq. (4) can be written as $\chi(\tau) = \exp(i\nu\tau)f(\tau)$, where $f(\tau)$ is a π-periodic function and ν is the Floquet exponent—both dependent on the parameters a and q. According to this solution, the amplitude of the magnetic field $x(t)$ evolves exponentially in time with the maximum growth rate $\gamma = (|\Im[\omega\nu]| - 1)/2$, where the modulus accounts for the time-reflection symmetry of Eq. (4) [26]. Thus, the amplitude of the magnetic field grows exponentially when $\gamma > 0$, whilst the marginal state is defined by $\gamma = 0$. We use the Maple computer algebra software to calculate Floquet exponent which defines the growth rate γ depending on the amplitude δ and the frequency ω. Next, we find the critical oscillation amplitude δ_c as the function of frequency ω for fixed values of the mean rotation α_0 by solving numerically equation $|\Im[\omega\nu]| = 1$, which corresponds to $\gamma = 0$. Eventually, we determine the minimal oscillation amplitude $\delta_{\min}$ and the corresponding frequency at which an exponentially growing magnetic field first appears. The corresponding numerical results are presented and discussed below.

3. Results

We start with a marginal mean disc rotation rate $\alpha_0 = 0$, which corresponds to the dynamo excitation threshold when the disc rotates steadily, i.e., $\gamma = 0$ when $\delta = 0$. As seen in Fig. 2(a), the disc oscillations about the critical rotation rate with a sufficiently small amplitude ($\delta = 0.25$) brings the growth rate to a constant level below zero as the frequency is reduced (reciprocal frequency increased). As the oscillation amplitude increases, firstly, the growth rate splits into separate frequency bands whose width decreases as $\sim 1/\omega$ for $\omega \to 0$. In order to show this increasingly fine-scale structure of the growth rate as $\omega \to 0$, we use the reciprocal frequency which is proportional to the period of disc oscillations. Secondly, the growth starts to increase with the oscillation amplitude and approaches zero again at $\delta \approx 0.75$ for a certain critical frequency. Further increase in the oscillation amplitude to $\delta = 1$ results in the appearance of several frequency bands with positive growth rates ($\gamma > 0$) which are shown as filled regions in Fig. 2(a). The critical amplitude δ_c, at which the growth rate turns zero, is shown for $\alpha_0 = 0$ in 2(b) against the oscillation frequency. Marginal state with $\gamma = 0$ corresponds to the boundaries of the separate frequency bands shown by different colors. Growth rate is positive corresponding to the dynamo action inside the filled frequency bands, which approach each other closely as the oscillation amplitude increases. Note that only a certain number of first instability bands are shown in Fig. 2(b). There is an infinite sequence of similar parametric instability bands of decreasing width as $\omega \to 0$, which is typical for Mathieu equation. Thus, the range