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The two-dimensional three-body problem in the large magnetic field limit is integrable

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ABSTRACT

The problem of N particles interacting through pairwise central forces is notoriously intractable for $N \geq 3$. Some remarkable specific cases have been solved in one dimension. Here we show that the guiding center approximation—valid for charges moving in two dimensions in the limit of large constant magnetic fields—simplifies the three-body problem for an arbitrary interparticle interaction invariant under rotations and translations, making it solvable by quadratures. A spinorial representation for the system is introduced, which allows a visualization of its phase space as the corresponding Bloch sphere. Finally, a discussion of the quantization of the problem is presented.

1. Introduction

It is only in a few select cases that the N-body problem, with $N \geq 3$, is known to be integrable. In arbitrary dimensions, the best known example is that of N particles interacting through linear forces, first solved by Newton [1]. In one dimension, there are several cases, such as that of N particles interacting through an r^{-2} potential. This was solved by Calogero [2] and Marchioro [3] for $N = 3$ (but see also [4] for earlier related results) and by Calogero [5] and Sutherland [6] for the case of the quantum system with arbitrary N and all interaction strengths equal; the corresponding classical problem was solved by Moser [7]. While integrable N-body problems can also be found in two and three dimensions, these remarkable results generally involve somewhat peculiar features, such as velocity-dependent forces, many-body interactions, or Hamiltonians that are not of the usual form of the sum of kinetic and potential energy. The reader will find an extensive treatment and many references in [8] and more recent results in [9].

The aim of this letter is to present a general class of integrable systems of a rather different nature. On the one hand, they admit a broad class of interactions between the three particles: any force defined by a rotationally and translationally invariant potential is allowed. This includes in particular the case in which the particles interact via arbitrary pairwise central potentials. On the other hand, they are explicitly limited to the case of three particles moving in two dimensions. The feature that makes the problem solvable is that the particles are charged with the same charge e in the presence of a strong constant magnetic field B. The latter induces a rapid circular motion of particle i of radius $r_i = m_i v_i/|eB|$ and frequency $\omega_c = |eB|/m_i$, where m_i and v_i, are the mass and speed of the particle respectively (we use $c = 1$ throughout). If the field is sufficiently strong, the r_i become negligible relative to any other length-scales of the problem. In this limit, the dynamics can be described, as is well-known, by an effective Hamiltonian system describing the secular motion of the center of the circular motion. In this paper, we show that this Hamiltonian is integrable by virtue of the geometric symmetries of the interaction potential. In that sense, the result is elementary, but also quite general.

In Section 2 we describe in detail the system to be studied. We first describe the full Hamiltonian, which is not solvable, and proceed to sketch the reduction process to the so-called guiding center Hamiltonian. The reduction is described in greater detail in Appendix A. In Section 3 we show how the system can be solved and introduce an appropriate set of variables to visualize the motion. In Section 4 we discuss in somewhat greater detail the specific case in which the interaction potential is a power-law in the interparticle distance. In Section 5 we discuss the corresponding quantum system, analyzing in somewhat greater detail

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the Coulomb case. Finally, in Section 6, we present some conclusions.

2. Description of the system

Let us turn to a detailed description of the system. Let $\vec{q}_i$ and $\vec{p}_i$ be the position and canonical momentum vectors of particle $i = 1, 2, 3$, with components $q_{i,\alpha}$ and $p_{i,\alpha}$ respectively ($\alpha = 1, 2$), and suppose the exact Hamiltonian of the system is

$$H(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{3} \frac{1}{2m_i} \left[\left(p_{i,1} - eB\, q_{i,2} \right)^2 + p_{i,2}^2 \right] + V\left(\vec{q}_1, \vec{q}_2, \vec{q}_3 \right) + \frac{K}{2} \sum_{i=1}^{3} |\vec{q}_i|^2, \tag{1}$$

where the interaction has the symmetry

$$V\left(\vec{q}_1, \vec{q}_2, \vec{q}_3 \right) = V\left(\mathcal{R}\vec{q}_1 + \vec{a}, \mathcal{R}\vec{q}_2 + \vec{a}, \mathcal{R}\vec{q}_3 + \vec{a} \right) \tag{2}$$

for arbitrary translations $\vec{a}$ and rotations $\mathcal{R}$ in the plane. A well-known transformation, described in detail in Appendix A (see also [10,11]), leads to new sets of canonical variables: the kinematical momenta $\vec{\pi}_i = m\vec{v}_i$, and the so-called guiding centers $\vec{R}_i = \vec{q}_i - \hat{z} \times \vec{\pi}_i/(eB)$, which have the following Poisson brackets

$$\{\pi_{i,\alpha}, \pi_{i,\beta}\} = \epsilon_{\alpha\beta}\delta_{ij} eB \tag{3a}$$

$$\{R_{i,\alpha}, R_{j,\beta}\} = -\epsilon_{\alpha\beta}\delta_{ij}(eB)^{-1} \tag{3b}$$

$$\{\pi_{i,\alpha}, R_{j,\beta}\} = 0 \tag{3c}$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric tensor in two dimensions with $\epsilon_{12} = 1$. As $|B|$ becomes large, the cyclotron radii $r_i = |\vec{\pi}_i|/|eB|$ become far smaller than the scale at which the potential varies, and the $\vec{\pi}_i$ and $\vec{R}_i$ decouple. The guiding center motion is then well described by the Hamiltonian

$$H_{\mathrm{gc}}(\underline{x}, \underline{y}) = V\left[(x_1, y_1), (x_2, y_2), (x_3, y_3) \right] + \frac{\Omega\left(|\underline{x}|^2 + |\underline{y}|^2 \right)}{2}, \tag{4}$$

where Ω is K multiplied by an appropriate constant and the vectors $\underline{x} = (x_1, x_2, x_3)$ and $\underline{y} = (y_1, y_2, y_3)$ are the x and y components of the guiding centers in units chosen so as to render them canonically conjugate:

$$\{x_i, y_j\} = \delta_{i,j}, \qquad \{x_i, x_j\} = \{y_i, y_j\} = 0. \tag{5}$$

In words, we can thus say that, in the large field limit, the full Hamiltonian (1) reduces to its interaction term, the effect of the magnetic field being to make the x and y coordinates of the system canonically conjugate.

3. Exact solvability of the reduced Hamiltonian

From the Poisson brackets (4), it is readily seen that

$$T_y = \sum_{i=1}^{3} x_i, \quad T_x = \sum_{i=1}^{3} y_i, \quad J = \frac{1}{2} \sum_{i=1}^{3} \left(x_i^2 + y_i^2 \right), \tag{6}$$

generate translations in y, translations in x, and rotations about the origin, all of which are symmetries of the interaction potential. Moreover, the harmonic external potential is proportional to J, which has vanishing Poisson bracket with the scalar $T_x^2 + T_y^2$. We thus find two independent integrals of the motion in involution, which for later convenience, can be traded for functions representing the orbital and spin angular momenta

$$L = \frac{1}{6}\left(T_x^2 + T_y^2 \right), \qquad S = J - L\,. \tag{7}$$

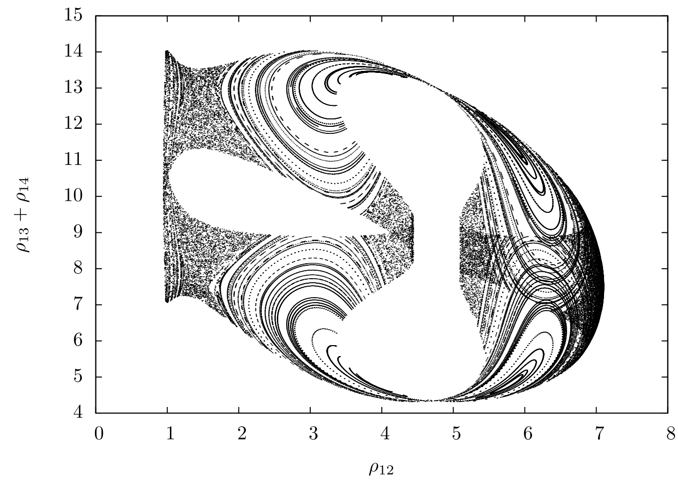

Fig. 1. Poincaré section of a four-particle system interacting via a $1/r^2$ potential. The total energy $E = 1$ and angular momentum $J = 5$; the Poincaré section is defined as those configurations in which the longest side of the triangle has squared length 7.8999396. The longest side is then defined to connect particles 3 and 4. ρ_{ij} is then defined to be the squared length between particles i and j. We notice a coexistence of smooth behavior (tori) with chaos.

The names total, orbital and spin angular momentum for J, L and S respectively are justified by the fact that these generate the corresponding types of rotations of the system. Since L, S and the Hamiltonian (4) are three integrals in involution, we conclude that the system is integrable.

We may thus set L and S to constant values and define coordinates describing the *shape* of the triangle formed by the three particles: these coordinates must then commute with S, since S generates global rotations of the triangle. Expressing the Hamiltonian in these coordinates leads to a problem in a two-dimensional phase space; that is, a one-dimensional problem, which can be straightforwardly solved by quadratures. Since the Hamiltonian only depends on the shape variables and on S, which is constant, one sees that the rotational motion only depends on the shape variables, and therefore can also be obtained by quadratures.

A remark concerning the symmetries of the Hamiltonian is in order: if the potential is not only symmetric under rotations, but also under some reflection, say under the transformation which changes the sign of all y_i but leaves the x_i invariant, then a form of *time-reversal invariance* is recovered, though it was, of course, broken in the original Hamiltonian. This occurs because, in the reduced system, such a reflection corresponds to an *anticanonical* involution, which plays the same role as time reversal. More concretely, this can be seen in the equations of motion, where such a reflection corresponds to changing the sign of t. Note that this happens for example, whenever the potential arises from two-body interactions.

Finally, the following question may, at this point, have occurred to the reader: is it possible that, for some choice of interaction potential, the integrability may hold for a larger number of particles? Two pieces of evidence make this unlikely: first we present a Poincaré section for 4 particles interacting via a $1/r^2$ potential (see Fig. 1). A chaotic region is readily apparent. A similar result was observed as well in the case of a $1/r$ interaction.

From an analytic viewpoint, the following argument serves to make the possibility of the four-particle system's being integrable rather unlikely. In the three-particle system, for every power law potential except the harmonic case, we have three stable and three unstable equilibria on the equator. The three unstable equilibria are, as usual, connected by separatrices. That is, the stable and unstable manifolds arising from these unstable fixed points join smoothly along the separatrix. However, as is well-known, see for