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State-independent error-disturbance trade-off for measurement operators

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ABSTRACT

In general, classical measurement statistics of a quantum measurement is disturbed by performing an additional incompatible quantum measurement beforehand. Using this observation, we introduce a state-independent definition of disturbance by relating it to the distinguishability problem between two classical statistical distributions – one resulting from a single quantum measurement and the other from a succession of two quantum measurements. Interestingly, we find an error-disturbance trade-off relation for any measurements in two-dimensional Hilbert space and for measurements with mutually unbiased bases in any finite-dimensional Hilbert space. This relation shows that error should be reduced to zero in order to minimize the sum of error and disturbance. We conjecture that a similar trade-off relation with a slightly relaxed definition of error can be generalized to any measurements in an arbitrary finite-dimensional Hilbert space.

1. Introduction

The uncertainty principle has been regarded as a fundamental principle in quantum mechanics. It asserts that we cannot get the precise values of two physical observables in a quantum state, unless they are compatible. The well-known version of this principle was formulated by Heisenberg in 1927, namely, [1]

$$\Delta x \Delta p \geq \hbar/2. \tag{1}$$

A more general form of it can be written as

$$\varepsilon(A)\eta(B) \geq \frac{|\langle\psi|[A,B]|\psi\rangle|}{2}, \tag{2}$$

where $\varepsilon(A)$ is the error with which the measurement of operator A is carried out, and $\eta(B)$ is the disturbance on the following measurement of operator B caused by the measurement of A. In Eq. (1), Δx and Δp can be interpreted as error and disturbance when A and B are position and momentum operators. Mathematically, Eq. (2) comes from the Robertson's uncertainty relation [2]:

$$\sigma(A)\sigma(B) \geq \frac{|\langle\psi|[A,B]|\psi\rangle|}{2}, \tag{3}$$

where $\sigma(X) = \sqrt{\langle\psi|X^2|\psi\rangle - \langle\psi|X|\psi\rangle^2}$ is the standard derivation of an observable X in a quantum state $|\psi\rangle$. Note that while Eq. (3), usually regarded as a rigorous version of Heisenberg's uncertainty principle [3–5], can be proven mathematically, the justification for relation Eq. (2) is currently on hot debate because additional conditions have been used in its derivation [6]. More importantly, several experiments showed that Eq. (2) is violated [7–9]. Thus, the trade-off relation that the higher the precision of measuring A, the stronger the disturbance on measuring B cannot be well-captured by Eq. (2).

Many important works in this area have been done, but the definitions of error and disturbance are still not settled [10]. Ozawa used the noise-operator based definition and proposed a "universally valid error-disturbance relation" [11]:

$$\varepsilon(A)\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{|\langle\psi|[A,B]|\psi\rangle|}{2}. \tag{4}$$

This uncertainty relation was later verified experimentally [7–9, 12–15] and inspired a lot of work on uncertainty relations [16–18], but some shortcomings were also pointed out [19,20]. For example, it seems to violate the proposed operational constraint that the error and disturbance should be non-zero if the outcome distribution is deviated from what is expected according to the Born rule [19].

Using distance between distributions is another way to quantify measurement errors [21–24], Busch et al. proved the original Heisenberg's error-disturbance relation in Eq. (2) by defining the error and disturbance as figures of merit characteristic of the

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measuring devices [21,25], which generated a debate over different approaches used in formalizing uncertainty relations [26,27].

In this paper, we introduce a straightforward definition of error and disturbance. The key observation is that given an arbitrary quantum state, the measurement statistics of a measurement operation on that state is unchanged if and only if we perform an additional compatible measurement to the state beforehand. Thus, we may define the disturbance of $\mathcal{B}$ (measurements of operator B) due to $\mathcal{A}$ (measurements of operator A) as the distance between the two probability distributions of the measurement outcomes due to $\mathcal{B}$ and $\mathcal{B}\circ\mathcal{A}$ maximized over all possible input quantum states. We introduce the definitions of error and disturbance in Sec. 2 and report a few basic properties of these quantities in Sec. 3. Then, in Sec. 4, we prove the error-disturbance trade-off relation for the case of 2-dimensional Hilbert space. In particular, we derive a sharp lower bound of the sum of error and disturbance. We also give the trade-off relation in d-dimensional Hilbert space for a special but important case. Finally, we draw a few conclusions in Sec. 5.

2. Definitions and notations

Suppose one is given a density matrix ρ in a d-dimensional Hilbert space with $d \geq 2$. Let $\mathcal{A}$ be the projective measurements of operator A with rank-one projectors. (Unless otherwise stated, all measurements in this paper are associated with rank-one projectors. Note that our discussion can be easily extended to the case of a general positive operator-valued measurement. We restrain from doing so to avoid unnecessary notational and indexing complications.) The probability distribution obtained from applying $\mathcal{A}$ to ρ is given by the vector

$$P_{\mathcal{A}}(\rho) = \left(p_i^{(\mathcal{A})}(\rho)\right)_{i=1}^{d} \equiv \left(\langle a_i|\,\rho\,|a_i\rangle\right)_{i=1}^{d}, \tag{5}$$

where $|a_i\rangle\langle a_i|$ is the rank-one projector corresponding to the ith measurement outcome. We now consider measuring ρ using another projective measurement $\mathcal{A}'$ before feeding the resultant state to $\mathcal{B}$. We write the probability distribution of the measurement outcomes of $\mathcal{A}'$ by $P_{\mathcal{A}'}(\rho)$. More importantly, the probability distribution of the final measurement outcomes of $\mathcal{B}\circ\mathcal{A}'$ is given by $P_{\mathcal{B}\circ\mathcal{A}'}(\rho) = P_{\mathcal{B}}(\rho')$ where $\rho' = \sum_i \langle a_i'|\,\rho\,|a_i'\rangle\,|a_i'\rangle\langle a_i'|$ with $|a_i'\rangle\langle a_i'|$ being the rank-one projector corresponding to the ith measurement outcome of $\mathcal{A}'$.

In general, $P_{\mathcal{B}\circ\mathcal{A}'}(\rho)$ is different from $P_{\mathcal{B}}(\rho)$ as measurements change the state of a quantum system. We would like to know how a change in measurement $\mathcal{A}'$ affects the change of $P_{\mathcal{B}}(\rho)$ through their classical statistics of their measurement outcomes only. With this motivation in mind, for any given metric $D(\cdot,\cdot)$ of an Euclidean space, we define the **state-dependent error** between $P_{\mathcal{A}}(\rho)$ and $P_{\mathcal{A}'}(\rho)$, and the **state-dependent disturbance** between $P_{\mathcal{B}}(\rho)$ and $P_{\mathcal{B}}(\rho')$ as

$$\varepsilon_\rho(\mathcal{A},\mathcal{A}') = D(P_{\mathcal{A}}(\rho), P_{\mathcal{A}'}(\rho)) \tag{6}$$

and

$$\eta_\rho(\mathcal{A}',\mathcal{B}) = D(P_{\mathcal{B}}(\rho), P_{\mathcal{B}}(\rho')), \tag{7}$$

respectively. Here, the definition of $\eta_\rho(\mathcal{A}',\mathcal{B})$ is known. Since our goal is to study the maximum pointwise deviation in the distribution of measurement outcomes, we use the metric based on the infinity norm, namely,

$$D(x, y) = \max_i |x_i - y_i|. \tag{8}$$

We now define the **state-independent error** and the **state-independent disturbance** by

$$\varepsilon(\mathcal{A},\mathcal{A}') = \max_\rho\ \varepsilon_\rho(\mathcal{A},\mathcal{A}') \tag{9}$$

and

$$\eta(\mathcal{A}',\mathcal{B}) = \max_\rho\ \eta_\rho(\mathcal{A}',\mathcal{B}), \tag{10}$$

where Eq. (10) measures the incompatibility of quantum measurements $\mathcal{A}'$ and $\mathcal{B}$, shown in [24]. From now on, the terms "error" and "disturbance" refer to the state-independent versions unless otherwise stated. Note that these definitions meet the proposed operational constraint [19] for $\varepsilon(\mathcal{A},\mathcal{A}') = 0$ if and only if $\mathcal{A} = \mathcal{A}'$ and $\eta(\mathcal{A}',\mathcal{B}) = 0$ if and only if $\mathcal{A}' = \mathcal{B}$.

Finally, to obtain a trade-off relation between error and disturbance in one measurement, that is, to find out how much we need to sacrifice on one to lower the other, just as what Heisenberg did, we introduce the **state-independent overall error**

$$\Delta(\mathcal{A},\mathcal{A}',\mathcal{B}) = \max_\rho \left(\varepsilon_\rho(\mathcal{A},\mathcal{A}') + \eta_\rho(\mathcal{A}',\mathcal{B})\right). \tag{11}$$

Clearly, $\varepsilon + \eta \geq \Delta$.

3. Basic properties of the state-independent error and disturbance

According to definitions in Sec. 2,

$$\varepsilon_\rho(\mathcal{A},\mathcal{A}') = \max_i \left|\mathrm{tr}\left(\rho(|a_i\rangle\langle a_i| - |a_i'\rangle\langle a_i'|)\right)\right| \tag{12}$$

$$\eta_\rho(\mathcal{A}',\mathcal{B}) = \max_i \left|\mathrm{tr}\Big(\rho\big(|b_i\rangle\langle b_i| - \sum_j \left|\langle a_j'|b_i\rangle\right|^2 |a_j'\rangle\langle a_j'|\big)\Big)\right|,$$

we have

$$\varepsilon(\mathcal{A},\mathcal{A}') = \max_i\ R\left(|a_i\rangle\langle a_i| - |a_i'\rangle\langle a_i'|\right) \tag{13}$$

and

$$\eta(\mathcal{A}',\mathcal{B}) = \max_i\ R\Big(|b_i\rangle\langle b_i| - \sum_k \left|\langle b_i|a_k'\rangle\right|^2 |a_k'\rangle\langle a_k'|\Big). \tag{14}$$

Here, $R(\cdot)$ is the spectral radius of a matrix (the largest of absolute values of the eigenvalues). Similarly, we have

$$\Delta(\mathcal{A},\mathcal{A}',\mathcal{B}) = \max_{i,j,\pm} R\Big(|a_i\rangle\langle a_i| - |a_i'\rangle\langle a_i'| \pm |b_j\rangle\langle b_j| \mp \sum_k \left|\langle b_j|a_k'\rangle\right|^2 |a_k'\rangle\langle a_k'|\Big). \tag{15}$$

Note that the maximum of ε_ρ can be attained by a pure state ρ; and similarly for η_ρ and $\varepsilon_\rho + \eta_\rho$.

Property 1 *(Range). The error satisfies*

$$\varepsilon(\mathcal{A},\mathcal{A}') \leq 1 \tag{16}$$

for any $\mathcal{A}, \mathcal{A}'$, with equality if and only if $\langle a_i'|a_i\rangle = 0$ for some i. In addition, the disturbance obeys

$$\eta(\mathcal{A}',\mathcal{B}) \leq 1 - 1/d; \tag{17}$$

for any $\mathcal{A}', \mathcal{B}$, with equality if and only if there is an i such that $|b_i\rangle$ is unbiased in $\left(|a_j'\rangle\right)_{j=1}^{d}$. That is to say, $\left|\langle a_j'|b_i\rangle\right|^2 = 1/d$ for all j.

Proof. The rank of the matrix $|a_i\rangle\langle a_i| - |a_i'\rangle\langle a_i'|$ is at most 2. Thus, the spectral radius of this matrix can be calculated easily as $|a_i\rangle$ and $|a_i'\rangle - \langle a_i|a_i'\rangle\,|a_i\rangle$ are orthogonal. Hence, Eq. (13) becomes

$$\varepsilon(\mathcal{A},\mathcal{A}') = \max_i \sqrt{1 - \left|\langle a_i'|a_i\rangle\right|^2}. \tag{18}$$

Consequently, $\varepsilon(\mathcal{A},\mathcal{A}') \leq 1$ with equality holds when there exists an i such that $\langle a_i'|a_i\rangle = 0$.