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Rank-dependant factorization of entanglement evolution

Michael Siomau [a,b,*]

[a] Physics Department, Jazan University, P.O. Box 114, 45142 Jazan, Kingdom of Saudi Arabia
[b] Network Dynamics, Max Planck Institute for Dynamics and Self-Organization (MPIDS), 37077 Göttingen, Germany

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ABSTRACT

The description of the entanglement evolution of a complex quantum system can be significantly simplified due to the symmetries of the initial state and the quantum channels, which simultaneously affect parts of the system. Using concurrence as the entanglement measure, we study the entanglement evolution of few qubit systems, when each of the qubits is affected by a local unital channel independently on the others. We found that for low-rank density matrices of the final quantum state, such complex entanglement dynamics can be completely described by a combination of independent factors representing the evolution of entanglement of the initial state, when just one of the qubits is affected by a local channel. We suggest necessary conditions for the rank of the density matrices to represent the entanglement evolution through the factors. Our finding is supported with analytical examples and numerical simulations.

1. Introduction

Current development of quantum technologies suggests quantum entanglement as the exclusive resource for many potential applications, such as quantum teleportation, superdense coding, quantum cryptography and quantum computing [1]. Apart from entanglement generation [2–4] and detection [5], successful practical utilization of entanglement-based quantum technologies demands efficient protocols for entanglement protection from detrimental environmental influence [6] and its recovery after a possible partial loss [7–9]. The construction of these protocols, in turn, requires exact methods for entanglement quantitative description as well as clear understanding of the fundamental laws of the entanglement evolution. The lack of an accurate entanglement measure for multipartite entangled systems [1] imposes serious limitations on our ability to describe complex entanglement dynamics. Therefore, any realistic situation, when the description of the entanglement evolution of a multipartite quantum system can be simplified, is of great practical importance.

An important example of such a simplified description of complex entanglement dynamics was given by Konrad et al. [10]. It was shown that the entanglement evolution of an arbitrary pure two-qubit state can be completely described by two factors, which are given by the initial entanglement of the pure state and the entanglement dynamics of the maximally entangled state. Subsequently, this result has been extended to the cases of high-dimensional bipartite systems [11–13], mixed initial states [14] and multiqubit systems [15]. But, so far the exact equations for the entanglement evolution has been obtained under assumption that just one subsystem of the entangled system undergoes the action of an environmental channel (i.e. the system is affected by a single-sided channel). If, in contrast, the system is subordinated to multi-sided channels, only inequalities can be derived theoretically. In practice, it is often required to distribute parts of an entangled system between several remote recipients [16]. In this case, each subsystem is coupled locally with some environmental channel, i.e. the quantum system is the subject of many-sided channels. Recently, we analyzed the entanglement dynamics of initially pure three-qubit Greenberger–Horne–Zeilinger (GHZ) state, when each qubit is simultaneously affected by a noisy channel [17]. We showed that, in some cases, the entanglement dynamics of the three-qubit system in many-sided channels can be completely described by factors, which represent the evolution of the entangled system in single-sided channels. Similar result has been independently obtained by Man et al. [18] for generalized multiqubit GHZ states. Moreover, using so-called G-concurrence [19] as the entanglement measure, Gheorghiu and Gour [20] have recently shown that the average loss of entanglement induced by many-sided local channels is independent on the initial state and is completely defined by the local channels.

In this paper, we analyze the entanglement evolution of two-, three- and four-qubit systems affected by local many-sided channels. Using Wootter's concurrence [21] for two qubits and its ex-

* Correspondence to: Physics Department, Jazan University, P.O. Box 114, 45142 Jazan, Kingdom of Saudi Arabia.
E-mail address: siomau@nld.ds.mpg.de.

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tension to higher dimensions [22] for multiqubit systems, we show that, for low-rank density matrices of the final state, such a complex entanglement dynamics can be completely described by factors representing single-sided entanglement dynamics of the initial state. For two qubits, in particular, the above factorization can be achieved, if the rank of the final state density matrix is two. For multiqubit systems, in contrast, the factorization is possible for density matrices with rank no higher than four. On analytical examples and by numerical simulations we show that the factorization is independent on the initial (pure) quantum state of the qubit system and the local channels, as far as the above rank conditions are fulfilled. Since it is generally difficult to generate a low rank final state density matrix out of an arbitrary initial multiqubit state, for three and four qubit systems, we shall assume first that the initial state is a maximally entangled state, either GHZ or W state. The symmetry of the initial states allows us generating final state density matrices with all possible ranks for arbitrary local channels. Later we shall relieve this latter assumption extending our results to the case of arbitrary initially pure state of the qubits.

This work is organized as follows. In the next section we shall briefly describe the entanglement measures of use and introduce the quantum operation formalism [23] that allows us to access the state dynamics of quantum systems under the action of local noisy channels. In Sec. 3 we step-by-step analyze the entanglement dynamics of two-, three- and four-qubit systems affected by local many-sided channels and show examples when such a complex entanglement dynamics can be factorized on terms representing single-sided entanglement evolution. In Sec. 4 we discuss possible implications of our results to theoretical and experimental description of the entanglement dynamics. We conclude in Sec. 5.

2. Concurrence and quantum state dynamics

2.1. The entanglement measure

It has been found difficult to quantify the entanglement of mixed many-partite states, and no general solution is known apart from few cases of low-dimensional systems [1]. Wootter's concurrence, for instance, allows us to compute the entanglement of an arbitrary state of a two-qubit system, which is given by the density matrix ρ, as $C_W = \max\{0, \lambda^1 - \lambda^2 - \lambda^3 - \lambda^4\}$. Here λ^i are the square roots of the four eigenvalues of the non-Hermitean matrix $\rho\,(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, if taken in decreasing order. It is important to note that this matrix is obtained from the density matrix ρ by simultaneous inversion of the single-qubit subsystems with the help of the only generator σ_y of the SO(2) group.

Various extensions of Wootter's concurrence have been worked out over the years [1]. Ou et al. [22], in particular, suggested a generalization of Wootter's concurrence for bipartite states, if the dimensions of the associated Hilbert subspaces are larger than two. For a $d_1 \otimes d_2$-dimensional quantum system, this concurrence can be written as

$$C = \sqrt{\sum_{m=1}^{d_1(d_1-1)/2} \sum_{n=1}^{d_2(d_2-1)/2} (C_{mn})^2}\,, \tag{1}$$

where each term C_{mn} is given by

$$C_{mn} = \max\{0, \lambda^1_{mn} - \lambda^2_{mn} - \lambda^3_{mn} - \lambda^4_{mn}\}\,. \tag{2}$$

Here, the λ^k_{mn}, $k = 1..4$ are the square roots of the four nonvanishing eigenvalues of the matrix $\rho\,\tilde{\rho}_{mn}$, if taken in decreasing order. These matrices $\rho\,\tilde{\rho}_{mn}$ are formed by means of the density matrix ρ and its complex conjugate ρ^*, and are further transformed by the operators $S_{mn} = L_m \otimes L_n$ as: $\tilde{\rho}_{mn} = S_{mn}\rho^* S_{mn}$. In this notation, moreover, L_m are $d_1(d_1-1)/2$ generators of the group SO(d_1), while the L_n are the $d_2(d_2-1)/2$ generators of the group SO(d_2).

Although the bipartite concurrence (1) reduces to Wootter's concurrence for the special case of two qubits, in general it is an approximate entanglement measure, which provides limited information about entanglement of the bipartite system [22]. While the dimensionality of the Hilbert space of a two-qubit system is four, the inversion of an arbitrary state ρ is unambiguously defined by the single generator of the SO(2) group. In higher dimensions, however, there is no unique way to invert a given quantum state [24]. Ambiguous choice of the state inversion leads to the summation over all possible $d_1(d_1-1)d_2(d_2-1)/4$ state inversions in Eq. (1) in all $2 \otimes 2$-dimensional subspaces of the original $d_1 \otimes d_2$-dimensional Hilbert state space of the bipartite system. The main consequence of such approximation for state inversion is that there may be only four nonzero eigenvalues of the matrix $\rho\,\tilde{\rho}_{mn}$, while the other $d_1 d_2 - 4$ eigenvalues of this matrix always vanish.

In spite of the above limitations, concurrence (1) has been shown to be quite powerful measure of entanglement [9,12,14]. Using the bipartite concurrence Li et al. [25] formulated an analytical lower bound for multiqubit concurrence, which is given by a squared sum of the bipartite concurrences computed for all possible bi-partitioning of the multiqubit system. For three qubits, in particular, the lower bound can be written in terms of the three bipartite concurrences that correspond to possible cuts of two qubits from the remaining one, i.e.

$$\tau_3(\rho) = \sqrt{\frac{1}{3}\left((C^{12|3})^2 + (C^{13|2})^2 + (C^{23|1})^2\right)}\,. \tag{3}$$

This lower bound, moreover, has been used to describe the entanglement dynamics of three-qubit states under the action of certain multi-sided noisy channels [26]. On particular analytical examples and by numerical simulations, it has been shown that for three-qubit density matrices with rank no higher than four, the lower bound (3) provides adequate description of the entanglement evolution irrespective from system–channel coupling rate and for all times of interaction. For density matrices with higher ranks, however, the lower bound vanishes after a finite time, while the quantum states it is applied to are not separable, i.e. possess certain amount of entanglement. This behavior of the lower bound (3) is not the consequence of the entanglement sudden death [27], but is induced by the approximate character of the bipartite concurrence (1) as an entanglement measure.

2.2. Quantum operation formalism

Quantum operation formalism is a very general and prominent tool to describe how a quantum system has been influenced by its environment. According to this formalism the final state of the quantum system, that is coupled to some environmental channel, can be obtained from its initial state with the help of (Kraus) operators

$$\rho_{\text{fin}} = \sum_i K_i\, \rho_{\text{ini}}\, K_i^\dagger\,, \tag{4}$$

and the condition $\sum_i K_i^\dagger K_i = I$ is fulfilled. Note that we consider only such system–environment interactions that can be associated with completely positive trace-preserving maps [23].

If the quantum system of interest consists of just a single qubit, which is subjected to some environmental channel A, then an arbitrary quantum operation associated with the channel's action can be expressed with the help of at most four operators [23]. Let us define the four operators through the Pauli matrices as

$$K_1(a_1) = \frac{a_1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ K_2(a_2) = \frac{a_2}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5}$$

$$K_3(a_3) = \frac{a_3}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ K_4(a_4) = \frac{a_4}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$