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Path integral in Snyder space

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ABSTRACT

The definition of path integrals in one- and two-dimensional Snyder space is discussed in detail both in the traditional setting and in the first-order formalism of Faddeev and Jackiw.

1. Introduction

The interest in noncommutative spaces has increased in recent years, because they may describe the structure of space (or spacetime) at the Planck scale, as several approaches to quantum gravity seem to indicate [1]. The formulation of quantum mechanics on a noncommutative space is usually called noncommutative quantum mechanics. Path integral techniques have demonstrated to be convenient in the study of this theory.

A characteristic of noncommutative spaces is that the corresponding classical phase space is not canonical, i.e. the Poisson brackets do not have the usual form. However, the standard definition of path integral assumes a canonical phase space [2–4], and one has therefore to extend the formalism to include this more general situation.

This is an interesting problem, that has been afforded in a variety of ways. In fact, several different approaches have been proposed for the definition of the path integral in noncommutative spaces. The first one is based on the noncanonical structure of the phase space: Darboux theorem ensures that it is always possible to find a transformation to canonical (and hence commutative) coordinates, that will deform the measure of the integral, but allow otherwise to use the standard formulation of the path integral [5–7]. A different approach uses the standard integration measure, but treats the products in the integrand as star products between functions of noncommutative coordinates [8]; this framework is more suitable for a generalization to field theory. Finally, some authors propose the adoption of smeared (coherent state) bases for the Hilbert space to avoid the use of noncommutative coordinates in the computation of the path integral [9,10]. Although we are not aware of any discussion in the literature, we presume that these approaches are equivalent.

Anyway, most work on the subject has been developed for the so-called Moyal plane [11], a simple model whose Poisson brackets are constant tensors, hence necessarily implying the breakdown of the Lorentz invariance. However, more general models of noncommutative spaces exist, in which the Lorentz invariance is preserved. The best known is the Snyder model [12], which, in spite of the presence in its definition of a parameter β with the dimension of inverse momentum, is Lorentz invariant. The quantum mechanics of the Snyder model has been studied in several papers [13,14].

In its nonrelativistic version, the Snyder model is based on a deformation of the Heisenberg algebra, given by the commutation relations

$$[q_i, p_j] = i(\delta_{ij} + \beta^2 p_i p_j), \qquad [q_i, q_j] = i\beta^2 J_{ij}, \qquad [p_i, p_j] = 0, \tag{1}$$

where q_i and p_i are the phase space coordinates, and J_{ij} the angular momentum generators; we use units in which $\hbar = 1$. Clearly, the classical limit of these commutators gives rise to a noncanonical phase space.

In this paper, we discuss the formulation of the one-particle nonrelativistic quantum mechanics of the Snyder model through path integral methods. We adopt the approach of [6] based on the noncanonical structure of phase space, since it is more suitable for our problem and closer to the spirit of Feynman's original idea.

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We give a detailed derivation starting from the definition of path integral and using a representation of the operators in terms of canonical coordinates. We also show that the same results can be recovered in a more formal way using the techniques introduced in [15] for the study of first-order systems, taking for granted the definition of path integral for canonical variables. Our work generalizes some results recently obtained in [16] for the case of one spatial dimension, correcting an error in the measure of the path integral proposed there.

2. Noncanonical classical mechanics

Before discussing the path integral formulation of Snyder quantum mechanics, we shortly review some facts concerning the noncanonical Hamiltonian formalism [17,18], that will be useful in the following.

Let us consider noncanonical fundamental Poisson brackets

$$\{\xi_i, \xi_j\} = \Omega_{ij}(\xi), \tag{2}$$

where ξ_i denotes the phase space variables q_i and p_i and Ω_{ij} is an invertible matrix. Then the Hamilton equations for the Hamiltonian $H(\xi)$ read

$$\dot{\xi}_i = \Omega_{ij}\frac{\partial H}{\partial \xi_j}, \tag{3}$$

or equivalently,

$$(\Omega^{-1})^{ij}\dot{\xi}_j = \frac{\partial H}{\partial \xi_i}. \tag{4}$$

We want to obtain these equation from the variation of a first-order action of the form

$$I = \int [a^i(\xi)\dot{\xi}_i - H(\xi)]dt. \tag{5}$$

Then one can easily check that the condition

$$\frac{\partial a^j}{\partial \xi_i} - \frac{\partial a^i}{\partial \xi_j} = (\Omega^{-1})^{ij} \tag{6}$$

must hold. Solving (6) for the a^i, one can write down the action which generates the Hamilton equations (3).

3. One-dimensional Snyder path integral

In this section we investigate the path integral for the one-dimensional Snyder model. Although in this case noncommutativity is of course absent, the symplectic structure is still noncanonical, and the discussion will be useful for the understanding of the higher-dimensional case.

Clearly, when investigating the Snyder model, one must use the phase space formulation of the path integral. For a particle satisfying canonical Poisson brackets, moving in a one-dimensional space, the path integral is defined as

$$A = \int \mathcal{D}p\mathcal{D}q\, \mathrm{e}^{iI}, \tag{7}$$

where

$$I = \int_{t_i}^{t_f} L dt = \int_{t_i}^{t_f} (p\dot{q} - H(q,p))dt \tag{8}$$

is the action (with L the Lagrangian and H the Hamiltonian), and $\mathcal{D}p\mathcal{D}q$ is a measure on the space of paths in phase space that will be defined below.

It can be shown that in a momentum basis the transition amplitude from an initial state of momentum p_i at time t_i to a final state of momentum p_f at time t_f is given by

$$< p_f|e^{-i\hat{H}(t_f-t_i)}|p_i> = A. \tag{9}$$

We have chosen a momentum basis, because, when we shall consider Snyder space, the standard position variables will not commute and hence do not form a complete set of observables.

We wish to generalize this formula to the one-dimensional Snyder phase space, whose only nontrivial Poisson bracket is

$$\{q, p\} = 1 + \beta^2 p^2. \tag{10}$$

Given the Hamiltonian $H = \frac{p^2}{2} + V(q)$, the Hamilton equations in Snyder space read

$$\dot{q} = (1+\beta^2p^2)p, \qquad \dot{p} = -(1+\beta^2p^2)\frac{\partial V}{\partial q}. \tag{11}$$

These equations can be obtained from an action principle, as discussed in section 2. Defining $\xi_1 = q$, $\xi_2 = p$, the inverse of the symplectic matrix associated to (10) will be

$$(\Omega^{-1})^{ij} = \frac{1}{1+\beta^2p^2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{12}$$

Inserting in (6) one can get the particular solution

$$a_1 = 0, \qquad a_2 = \frac{-q}{1+\beta^2p^2}, \tag{13}$$

from which follows the action

$$I = \int \left(-\frac{q\dot{p}}{1+\beta^2p^2} - H\right)dt = \int \left(\frac{\arctan\beta p}{\beta}\,\dot{q} - H\right)dt, \tag{14}$$

where the two expressions are related by an integration by parts. We will now show that inserting (14) into (7) gives the correct expression for the path integral.

We first recall some results concerning the quantum mechanics of the one-dimensional Snyder model [19]. The Poisson bracket (10) goes into the commutator

$$[\hat{q}, \hat{p}] = i(1+\beta^2\hat{p}^2). \tag{15}$$

The operators $\hat{q}$ and $\hat{p}$ obeying (15) can be represented in a momentum basis by [20]

$$\hat{p} = p, \qquad \hat{q} = i(1+\beta^2p^2)\frac{\partial}{\partial p}. \tag{16}$$

These operators are hermitian with respect to the scalar product

$$<\psi\,|\,\phi> = \int_{-\infty}^{+\infty} \frac{dp}{1+\beta^2p^2}\,\psi^*(p)\phi(p). \tag{17}$$

The identity operator can therefore be expanded in terms of momentum eigenstates $|\,p>$ as [19]

$$1 = \int_{-\infty}^{\infty} \frac{dp}{1+\beta^2p^2}\,|\,p><p\,|, \qquad \text{with} \quad <p\,|\,p'> = (1+\beta^2p^2)\delta(p-p'). \tag{18}$$

The eigenvalue equation for the position operator, $\hat{q}\,|q> = q\,|q>$, has formal solutions[1]

$$<p\,|\,q> \propto \mathrm{e}^{-iq\frac{\arctan\beta p}{\beta}} \tag{19}$$

[1] These eigenstates are not physical, because they have infinite energy [19], but are sufficiently regular to adopt them in this setting.