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# Delay-dependent exponential stability for neural networks with discrete and distributed time-varying delays ☆

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ABSTRACT

This Letter studies the exponential stability for a class of neural networks (NNs) with both discrete and distributed time-varying delays. Under weaker assumptions on the activation functions, by defining a more general type of Lyapunov functionals and developing a new convex combination technique, new less conservative and less complex stability criteria are established to guarantee the global exponential stability of the discussed NNs. The obtained conditions are dependent on both discrete and distributed delays, are expressed in terms of linear matrix inequalities (LMIs), and contain fewer decision variables. Numerical examples are given to illustrate the effectiveness and the less conservatism of the proposed conditions.



## 1. Introduction

Now, neural networks (NNs) are widely studied, because of their immense potentials of application prospective in a variety of areas, such as signal processing, pattern recognition, static image processing, associative memory, and combinatorial optimization. In order to deal the moving images processing, delayed neural networks were introduced [1]. Due to the finite speed of information processing, the existence of time delays frequently causes oscillation, divergence, or instability in NNs. Therefore, the stability problem of delayed neural networks has become a topic of great theoretic and practical importance in recent years [2–10].

NNs usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths [11]. Thus, there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays [12]. Recently, there has been a growing interest in the study of neural networks with discrete and distributed delays. In [13], NNs with discrete and distributed constant delays was investigated, several sufficient conditions for ensuring the existence and global asymptotical stability of the equilibrium point were derived, and these results were further extended to uncertain NNs in [14].

For neural networks with discrete and distributed time-varying delays, [15] discussed the robust asymptotic stability and the constraint $\dot{d}(t) < 1$ on the discrete time-varying delay was relaxed by using Lyapunov theory and Leibniz–Newton formula. However, the activation functions in [15] were assumed to be monotonically nondecreasing. [16] assumed the activation functions to satisfy a global Lipschtiz condition, which implies that the activation functions are not necessary to be monotonically nondecreasing, and a delay-dependent exponential stability condition was derived without considering the differentiability of delays. [17] proposed an exponential stability criterion by constructing an augmented Lyapunov functional, where the discrete delay $d(t)$ must be differentiable and $\dot{d}(t) < 1$. Clearly, such constraints on the delay term $d(t)$ and the activation functions were relatively strong. In addition, it should be pointed out that the stability results given in [15,16] and [17] were conservative in some extent, and it leaves some room for further improvement.

In this Letter, the exponential stability for a class of NNs with both discrete and distributed time-varying delays is also investigated. Unlike the existing works, a new convex combination technique is developed based on the inequality $\frac{1}{d(t)} + \frac{1}{d-d(t)} \geqslant \frac{4}{d}$

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$(0 \leqslant d(t) \leqslant d)$. Combining with the defining of new Lyapunov functionals and a delay decomposition method, new delay-dependent exponential stability criteria are derived in terms of linear matrix inequalities (LMIs). Meanwhile, the activation functions here are assumed to satisfy a sector bound condition, so the considered NNs are more general since the activation functions may be neither monotonic nor differentiable. It is shown that the newly obtained results are less conservative and more applicable than the existing corresponding ones. Since fewer decision variables are involved, the newly obtained results are also less complex. Some numerical examples will be given to show the effectiveness of the main results.

*Notation*: Throughout this Letter, a real symmetric matrix $P > 0 (\geqslant 0)$ denotes $P$ being a positive definite (positive semi-definite) matrix, and $A > B (A \geqslant B)$ means $A - B > 0 (\geqslant 0)$. $\| \cdot \|$ is the Euclidean norm in $R^n$. If $A$ is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup_{\|x\|=1}\{\|Ax\|\} = \sqrt{\lambda_{\max}(A^T A)}$, where $\lambda_{\max}(A)$ (respectively, $\lambda_{\min}(A)$) means the maximum (respectively, minimum) eigenvalue of $A$. $I$ is used to denote an identity matrix with proper dimension. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The symmetric terms in a symmetric matrix are denoted by $*$.

## 2. Problem formulation and preliminaries

Consider the following neural network with both discrete and distributed delays:

$$\dot{x}(t) = -Cx(t) + Af\big(x(t)\big) + Bf\big(x\big(t - d(t)\big)\big) + D\int_{t-\tau(t)}^{t} f\big(x(s)\big)\,ds + J, \quad t \geqslant 0, \tag{1}$$

$$x(t) = \phi(t), \quad t \in [-h, 0], \tag{2}$$

where $x(\cdot) = [x_1(\cdot), x_2(\cdot), \ldots, x_n(\cdot)]^T \in R^n$ is the neuron state vector, $f(x(\cdot)) = [f_1(x_1(\cdot)), f_2(x_2(\cdot)), \ldots, f_n(x_n(\cdot))]^T \in R^n$ denotes the neuron activation function, and $J = [J_1, J_2, \ldots, J_n]^T \in R^n$ is a constant external input vector. $C = \mathrm{diag}\{c_1, \ldots, c_n\}$ with $c_i > 0$ $(i = 1, 2, \ldots, n)$, and $A, B, D$ are the connection weight matrix, the discretely delayed connection weight matrix and the distributively delayed connection weight matrix, respectively. $d(t)$ and $\tau(t)$ denote the discrete time-varying delay and the distributed time-varying delay, respectively, and are assumed to satisfy $0 \leqslant d(t) \leqslant d$, $0 \leqslant \tau(t) \leqslant \tau$, where $d$ and $\tau$ are positive constants. The initial vector $\phi(t)$ is bounded and continuously differential on $[-h, 0]$, where $h = \max\{d, \tau\}$.

Similar to [19], the following assumptions will be made throughout the Letter.

(H1). The activation functions $f_i(\cdot)$ $(i = 1, 2, \ldots, n)$ are bounded.
(H2). There exist some constants $l_i^-, l_i^+$ $(i = 1, 2, \ldots, n)$ such that

$$l_i^- \leqslant \frac{f_i(x) - f_i(y)}{x - y} \leqslant l_i^+, \quad \forall x, y \in R,\ x \neq y. \tag{3}$$

**Remark 1.** The above assumption (H2) on the activation function was originally proposed in [18,19], and widely used in many papers. In [15], the activation functions $f_i(\cdot)$ $(i = 1, 2, \ldots, n)$ are required to satisfy $f_i(0) = 0$ and $l_i^- = 0$, while the activation functions $f_i(\cdot)$ $(i = 1, 2, \ldots, n)$ satisfy global Lipschitz conditions in [16]. It is clear that the activation functions satisfying the sector bound condition (3) is more general than the corresponding ones in [15] and [16].

Assume that $x^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ is an equilibrium point of system (1), by choosing the coordinate transformation $z(\cdot) = x(\cdot) - x^*$, (1) is changed into the following error system

$$\dot{z}(t) = -Cz(t) + Ag\big(z(t)\big) + Bg\big(z\big(t - d(t)\big)\big) + D\int_{t-\tau(t)}^{t} g\big(z(s)\big)\,ds, \tag{4}$$

where $z(\cdot) = [z_1(\cdot), z_2(\cdot), \ldots, z_n(\cdot)]^T$ is the state vector of the transformed system, $g(z) = [g_1(z_1(\cdot)), g_2(z_2(\cdot)), \ldots, g_n(z_n(\cdot))]^T$ and $g_i(z_i(\cdot)) = f_i(z_i(\cdot) + x_i^*) - f_i(x_i^*)$ $(i = 1, 2, \ldots, n)$. Then, the functions $g_i(\cdot)$ $(i = 1, 2, \ldots, n)$ satisfy the following condition:

$$l_i^- \leqslant \frac{g_i(z_i)}{z_i} \leqslant l_i^+, \quad g_i(0) = 0,\ \forall z_i \neq 0. \tag{5}$$

In this Letter, we analyze the stability of system (4)–(5), and new global exponential stability criteria which are less conservative and less complex than the existing ones will be proposed. Undoubtedly, the newly proposed stability criterion will be more applicable since the activation functions are allowed to be more general.

To obtain our main results, the following definition and lemma are necessary.

**Definition 1.** The equilibrium point 0 of system (4) is said to be globally exponentially stable, if there exist scalars $k > 0$ and $\beta > 0$ such that

$$\big\|z(t)\big\| \leqslant \beta e^{-kt} \sup_{-h \leqslant s \leqslant 0} \big\|z(s)\big\|, \quad \forall t > 0, \tag{6}$$

and $k$ is called the exponential convergence rate index.

**Lemma 1.** *The following inequalities are true*:

$$0 \leqslant \int_0^{z_i(t)} \big(g_i(s) - l_i^- s\big)\,ds \leqslant \big(g_i\big(z_i(t)\big) - l_i^- z_i(t)\big)z_i(t), \tag{7}$$

$$0 \leqslant \int_0^{z_i(t)} \big(l_i^+ s - g_i(s)\big)\,ds \leqslant \big(l_i^+ z_i(t) - g_i\big(z_i(t)\big)\big)z_i(t). \tag{8}$$

**Proof.** From (5), it yields that

$$0 \leqslant \frac{g_i(s) - l_i^- s}{s}, \qquad 0 \leqslant \frac{l_i^+ s - g_i(s)}{s}, \quad \forall s \neq 0,$$

this implies that

$$0 \leqslant \int_0^{z_i(t)} \big(g_i(s) - l_i^- s\big)\,ds, \qquad 0 \leqslant \int_0^{z_i(t)} \big(l_i^+ s - g_i(s)\big)\,ds.$$

From (3), for any $y \neq s$, it gets that

$$\frac{(g_i(y) - l_i^- y) - (g_i(s) - l_i^- s)}{y - s} = \frac{(f_i(y + x_i^*) - f_i(s + x_i^*)) - l_i^-(y - s)}{y - s} \geqslant 0,$$

this implies that $g_i(y) - l_i^- y$ is monotonically nondecreasing respect to $y$. So, the right inequality in (7) is true.

Similarly, $l_i^+ y - g_i(y)$ is also monotonically nondecreasing respect to $y$ and the right inequality in (8) is true.

This completes the proof. □