Physics Letters A 380 (2016) 1593–1599

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Weak measurements measure probability amplitudes (and very little else)

D. Sokolovski [a,b,*]

[a] *Departmento de Química-Física, Universidad del País Vasco, UPV/EHU, Leioa, Spain*
[b] *IKERBASQUE, Basque Foundation for Science, Maria Diaz de Haro 3, 48013, Bilbao, Spain*

A R T I C L E I N F O

Article history:
Received 16 November 2015
Received in revised form 25 February 2016
Accepted 29 February 2016
Available online 2 March 2016
Communicated by P.R. Holland

Keywords:
Feynman paths
Quantum measurements
Apparent "superluminality"

A B S T R A C T

Conventional quantum mechanics describes a pre- and post-selected system in terms of virtual (Feynman) paths via which the final state can be reached. In the absence of probabilities, a weak measurement (WM) determines the probability amplitudes for the paths involved. The weak values (WV) can be identified with these amplitudes, or their linear combinations. This allows us to explain the "unusual" properties of the WV, and avoid the "paradoxes" often associated with the WM.

1. Introduction

Ever since its inception in 1988 [1] the subject of the so-called quantum weak values (WV) remained a controversial topic (for early critique see [2,3]). In more recent developments, the authors of [4] put forward a much debated [5,6] proposal for generalising the WV to classical theories, while Steinberg [7] suggested the use of weak measurements (WM) for probing certain "surreal" elements of quantum physics. A few years ago, the authors of [8] have demonstrated experimentally how WM can be used to (indirectly) measure the system's wave function. One might feel that a clarification of what actually happens in a WM is in order, and the purpose of this paper is to provide one based on the concepts conventionally used in quantum theory.

The history of the WV goes back to Feynman, who used the mean value of a functional, averaged with the *probability amplitudes*, to illustrate certain aspects of quantum motion [9]. Feynman averages naturally arise, for example, in an attempt to measure the time spent by a tunnelling particle in the barrier [10]. The WM, designed to perturb the measured system as little as possible, were later studied in terms of Krauss operators and the POVM's, and found applications in the analysis of continuous measurements [11–13]. The subject gained in popularity when the authors of [1] pointed out certain "unusual" properties of the WV. "WM elements of reality" were introduced in [14], which elevated the discussion to a yet higher philosophical level. Subsequent attempts to better understand the properties of the WV were made, e.g., in the analysis of the "complex probabilities" in [15]. General reviews of the subject can be found, for example in Refs. [16–18]. In a more recent review of the practical aspects of WV [19] the authors characterised the WV as *"complex numbers that one can assign to the powers of a quantum observable operator $\hat{A}$ using two states, an initial state $|i\rangle$..., and a final state $|f\rangle$..."* This still leaves open the original question posed by the authors of [1]: what, if anything, the WV tell us about the intermediate state of a pre- and post-selected system?

We will answer it in the following way: in the case of intermediate measurements made on a pre- and post-selected system one must consider the system's histories referring to at least three different moments of time. Such histories, in general, interfere, and are conventionally characterised by probability amplitudes [20]. A WM destroys coherence between the histories only slightly and, in the absence of probabilities, measures the corresponding probability amplitudes or, more generally, various combinations of its real and imaginary parts. We will show that this simple observation allows one to avoid the notions of "anomalous" weak values [1,4], quantum system "being at two different places at the same time" [21], "photons disembodied from its polarisation" [22,23], or violation of Einstein's causality in classically forbidden transitions [24]. For consistency, we will need to reproduce some of the known results, and we will try to do it in the briefest possible manner in the following Sections.

* Correspondence to: Departmento de Química-Física, Universidad del País Vasco, UPV/EHU, Leioa, Bizkaia, Spain.
E-mail address: dgsokol15@gmail.com.

http://dx.doi.org/10.1016/j.physleta.2016.02.051
0375-9601/

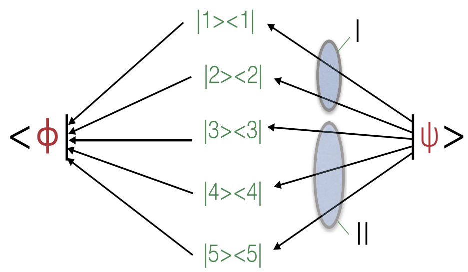

Fig. 1. (Colour online.) A system in a 5 dimensional Hilbert space can reach the final state $|\phi\rangle$ via five virtual paths $\{i\}$ with probability amplitudes $A_i^{\phi\leftarrow\psi}$. An accurate measurement of an operator $\hat{S}=\sum_{i=1}^{2}|i\rangle\langle i|-\sum_{i=3}^{5}|i\rangle\langle i|$ with degenerate eigenvalues of 1 and -1 creates two real pathways $I=\{1+2\}$ and $II=\{3+4+5\}$, travelled with the probabilities ω_I and ω_{II} given by Eq. (11). A WM of $\hat{S}$ determines the difference between relative amplitudes for the virtual paths I and II in Eq. (17), $\alpha_I-\alpha_{II}$.

2. Paths, amplitudes, and meters

Following [1] we consider a system in a N-dimensional Hilbert space with a Hamiltonian $\hat{H}$. We also consider an arbitrary operator $\hat{S}$, with the eigenvalues S_i and the eigenstates $|i\rangle$, $i=1,2,\ldots,N$. At $t=0$ the system is prepared (pre-selected) in a state $|\psi\rangle$ and at $t=T$ we check if the system is (post-select the system) in another state $|\phi\rangle$. If it is, we will keep the results of all other measurements we may make halfway into the transition, at $t=T/2$. The probability amplitude for a successful post-selection is then $A^{\phi\leftarrow\psi}=\langle\phi|\exp(-i\hat{H}T)|\psi\rangle$. Inserting the unity $\sum_i|i\rangle\langle i|=1$ at $t=T/2$ we have

$$A^{\phi\leftarrow\psi}=\sum_{i=1}^{N}A_i^{\phi\leftarrow\psi},$$
$$A_i^{\phi\leftarrow\psi}\equiv\langle\phi|\exp(-i\hat{H}T/2)|i\rangle\langle i|exp(-i\hat{H}T/2)|\psi\rangle \tag{1}$$

This can be seen as a variant of the most basic quantum mechanical problem [20]: a system may reach the final state from the initial state via N paths (see Fig. 1). The paths are determined by the nature of the quantity $\hat{S}$, and their amplitudes depend on $\hat{S}$, as well as on the initial and final states $|\psi\rangle$ and $|\phi\rangle$.

The paths may be either interfering or exclusive alternatives [20], depending on what is done at $t=T/2$. If nothing is done, N *virtual* paths form a single route, and their amplitudes should be added as in Eq. (1) [20]. The probability to arrive in ϕ is then given by $P^{\phi\leftarrow\psi}=|\sum_{i=1}^{N}A_i^{\phi\leftarrow\psi}|^2$. Alternatively, an external meter can destroy interference between the paths. If the destruction is complete, the paths become *real* and can be equipped with probabilities $|A_i^{\phi\leftarrow\psi}|^2$. The probability of a successful post-selection is now given by $P^{\phi\leftarrow\psi}=\sum_{i=1}^{N}|A_i^{\phi\leftarrow\psi}|^2$.

3. Von Neumann measurements with post-selection

To see how interference between the paths shown in Fig. 1 can be destroyed, we employ a von Neumann pointer with the position f and the momentum λ, briefly coupled to the system around $t=T/2$ via an interaction Hamiltonian $-i\delta(t-T/2)\partial_f\hat{S}$ (we use $\hbar=1$). The meter is prepared in a state $|M\rangle$, such that $G(f)\equiv\langle f|M\rangle$ is a real function which peaks around the origin $f=0$ with a width Δf,

$$G(f)=\langle f|M\rangle=(\Delta f)^{-1/2}G_0(f/\Delta f), \tag{2}$$

where $G_0(f)=G_0(-f)$, $G_0(f)_{|f|\to\infty}\to 0$ and $\int G_0^2(f)df=1$. After a successful post-selection, the meter is in a pure state $|M'\rangle$ (the result is well known, see, for example, [1])

$$G'(f)=\langle f|M'\rangle=\sum_{i=1}^{N}A_i^{\phi\leftarrow\psi}G(f-S_i). \tag{3}$$

In the momentum space, the meter's final state is given by

$$G'(\lambda)=\langle\lambda|M'\rangle=G(\lambda)\sum_{i=1}^{N}A_i^{\phi\leftarrow\psi}\exp(-i\lambda S_i), \tag{4}$$

where $G(f)=(2\pi)^{-1/2}\int G(\lambda)\exp(i\lambda f)d\lambda$. Repeating the experiment many times we can evaluate the mean pointer position or the momentum after the measurement,

$$\langle f\rangle_{\hat{S}}=\int f|G(f)|^2df\Big/\int|G(f)|^2df, \tag{5}$$

and

$$\langle\lambda\rangle_{\hat{S}}=\int\lambda|G(\lambda)|^2d\lambda\Big/\int|G(\lambda)|^2d\lambda. \tag{6}$$

So what can be learnt about the condition of a pre- and post-selected system at $t=T/2$? It is convenient to write the operator $\hat{S}$ as a sum of projectors on its eigenstates,

$$\hat{S}=\sum_{i=1}^{N}S_i\hat{P}_i,\quad \hat{P}_i\equiv|i\rangle\langle i|, \tag{7}$$

and consider the measurement of a $\hat{P}_i$ for various values of Δf.

4. Accurate (strong) measurements

Consider first an accurate (strong) measurement of a $\hat{P}_i$. Since Δf determines the uncertainty in the initial setting of the pointer, an accurate measurement would require $\Delta f\to 0$. If so, we easily find that

$$\langle f\rangle_i^{strong}=|A_i^{\phi\leftarrow\psi}|^2/|\sum_{i'=1}^{N}|A_{i'}^{\phi\leftarrow\psi}|^2\equiv\omega_i. \tag{8}$$

Thus, an accurate meter completely destroys the coherence between the paths in Fig. 1. Moreover, the measured mean value of the projector $\hat{P}_i$ gives the *relative frequency* with which the real path passing through the i-th state is travelled if the experiment is repeated many times. It is a simple matter to verify that for an arbitrary operator $\hat{S}$ with non-degenerate eigenvalues, $S_i\neq S_j$, the mean value of the pointer position gives the weighted sum of its eigenvalues,

$$\langle f\rangle_{\hat{S}}^{strong}=\sum_{i=1}^{N}\omega_i S_i. \tag{9}$$

This has an obvious classical meaning: if the value of the quantity $\hat{S}$ on the i-th path is S_i, and the i-th path is travelled with the probability ω_i, then the average value over many trials is given by the sum (9).

If K and $(N-K)$ eigenvalues of the measured $\hat{S}$ are degenerate, e.g., $S_1=\ldots=S_K\equiv S_I$, $S_{K+1}=\ldots=S_N\equiv S_{II}$, Eq. (3) shows that the interference between the paths within each group of eigenvalues is not destroyed by a strong measurement (SM) of $\hat{S}$ (see Fig. 1). Rather, in accordance with the Uncertainty Principle [20,25,26] they are combined into two real routes, with amplitudes,

$$A_I^{\phi\leftarrow\psi}=\sum_{i=1}^{K}A_i^{\phi\leftarrow\psi},\quad\text{and}\quad A_{II}^{\phi\leftarrow\psi}=\sum_{i=K+1}^{N}A_i^{\phi\leftarrow\psi}, \tag{10}$$

which are travelled with the probabilities