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Entanglement transformation between two-qubit mixed states by LOCC

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ABSTRACT

Based on a new set of entanglement monotones of two-qubit pure states, we give sufficient and necessary conditions that one two-qubit mixed state is transformed into another one by local operations and classical communication (LOCC). This result can be viewed as a generalization of Nielsen's theorem Nielsen (1999) [1]. However, we find that it is more difficult to manipulate the entanglement transformation between single copy of two-qubit mixed states than to do between single copy of two-qubit pure ones.

1. Introduction

Entanglement has important applications in quantum information theory [2]. In particular, shared bipartite entanglement is an essential resource for processing and transmitting quantum information. Since most applications of quantum information theory require the maximally entangled state in order to faithfully transmit quantum information, it is vital necessary to develop the technique of entanglement manipulation by LOCC to produce states with the amount of entanglement as great as possible from partially entangled states [3]. There has been much attention recently concerning entanglement manipulation of a single copy of pure states. Nielsen provided the necessary and sufficient conditions for pure bipartite states entanglement transformation by LOCC [1]. According to Nielsen's theorem the transformation between pure bipartite states $|\phi\rangle \xrightarrow{\text{LOCC}} |\psi\rangle$ can be performed by LOCC if and only if $\lambda_\phi \prec \lambda_\psi$, where λ_ϕ denotes the vector of Schmidt coefficients of $\mathrm{tr}_A(|\phi\rangle\langle\phi|)$ arranged in decreasing order. This result has been extended to the case where the probabilistic transformation $|\phi\rangle \xrightarrow{\text{LOCC}} \{p_i, |\psi_i\rangle\}$ can be accomplished iff $\lambda_\phi \prec \sum_i p_i \lambda_{\psi_i}$ and therefore majorization has received renewed attention in quantum information theory [4]. Obviously, for a pure bipartite state the entanglement is completely determined by its Schmidt coefficients according to the Schmidt decomposition [5]. Therefore, the Von Neumann entropy $S(\rho) = -\mathrm{tr}(\rho\log\rho)$ has been accepted as a canonical measure of pure bipartite states entanglement.

However, in practical application people would have to deal with mixed states rather than pure ones due to decoherence. Hence, it is extremely important to quantify mixed states entanglement for quantum information processing. This begs the question: Is majorization a suitable tool for transformation from one mixed state into another one by LOCC yet? If it is not true, what is the condition of $\rho \xrightarrow{\text{LOCC}} \sigma$? A function $f : \mathbf{R}^d \to \mathbf{R}$ is said to be Schur-convex if $x \prec y \Rightarrow f(x) \leqslant f(y)$ [6]. According to this definition the Von Neumann entropy is Schur-concave, that is, $\lambda_\rho \prec \lambda_\sigma \Rightarrow S(\rho) \geqslant S(\sigma)$. However, the Schumucher noiseless coding theorem for quantum information [7] implies that the maximum rate of error-free quantum information transmission is $S(\rho)$ qubits per signal. If $S(\rho) \geqslant S(\sigma)$ holds, then the coding theorem will be violated clearly [8]. So we have to develop other measures to quantify the entanglement of bipartite mixed states. Gour considered an essentially different type of measure of entanglement for two-qubit quantum systems [9]. For any two-qubit pure state $|\varphi\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$ the author defined the entanglement as follows: $E_\mu(|\varphi\rangle) = f_\mu(x) = \begin{cases} \frac{x}{\mu} & \text{for } x \leqslant \mu \\ 1 & \text{for } x > \mu \end{cases}$ with $\forall 0 < \mu \leqslant 1$, where $x = 2\min\{\lambda_0, \lambda_1\}$. Based on this definition, Gour gave nec-

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essary and sufficient conditions for entanglement transformation between two probability distribution of two-qubit pure states by LOCC. However, this result cannot be viewed as the entanglement transformation between two-qubit mixed states by LOCC. For example, let $\rho = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-|$ and $\sigma = |\Phi^+\rangle\langle\Phi^+|$ with $|\Phi^\pm\rangle = \frac{|00\rangle\pm|11\rangle}{\sqrt{2}}$. It is well known that ρ cannot be transformed to σ by LOCC, although they still meet the condition mentioned in Ref. [9]. In this Letter we first define a new set of entanglement monotones of two-qubit pure states. The necessary and sufficient conditions for entanglement transformation between two-qubit mixed states by LOCC is then represented.

2. The condition for entanglement transformation between two-qubit mixed states

We first briefly summarize some basic concepts and results that are needed for further treatment. An ensemble of pure states, which is usually represented by $\{p_i, |\psi_i\rangle\}$, is characterized by a finite set of positive numbers p_i ($\sum_i p_i = 1$) and a corresponding set of normalized vectors $|\psi_i\rangle$ of the Hilbert space $\mathcal{H}$ [10]. The density operator ρ (a trace one, semi-definite positive operator) associated to $\{p_i, |\psi_i\rangle\}$ is defined as:

$$\rho = \sum_{i=1} p_i|\psi_i\rangle\langle\psi_i|, \quad \sum_{i=1} p_i = 1,\ p_i \geqslant 0. \tag{1}$$

Following the ideas presented in Ref. [9], we first discuss the entanglement transformation between two ensembles $\{p_i; |\psi_i\rangle\} \xrightarrow{\text{LOCC}} \{q_j; |\phi_j\rangle\}$. Suppose Alice and Bob share a pure state $|\psi\rangle$, and then perform a quantum operation ε which outputs the pure states $|\psi_i\rangle$ $(i = 1, 2, \ldots, n)$ with probability p_i. We then consider that Alice and Bob obtain an ensemble $\{p_i, |\psi_i\rangle\}$. Alice and Bob perform again quantum operations ε'_i which output the pure states $|\phi_j\rangle$ with conditional probability p_{ji} for all the possible outcome states, that is, $\varepsilon'_i(|\psi_i\rangle\langle\psi_i|) = \sum_j p_{ji}|\phi_j\rangle\langle\phi_j|$. Thus, the transformation $\varepsilon' = (\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_n)$, defined between the ensembles $\{p_i, |\psi_i\rangle\}$ and $\{q_j, |\phi_j\rangle\}$, outputs the states $|\phi_j\rangle$ with probability $q_j = \sum_i p_i p_{ji}$, that is, $\varepsilon'(\sum_i p_i|\psi_i\rangle\langle\psi_i|) = \sum_{i,j} p_i p_{ji}|\phi_j\rangle\langle\phi_j|$ with $q_j = \sum_i p_i p_{ji}$, where $\sum_j p_{ji} = 1$.

We write an arbitrary two-qubit pure state $|\psi\rangle$ as $|\psi\rangle = \sqrt{x}|00\rangle + \sqrt{1-x}|11\rangle$ with $x \leqslant \frac{1}{2}$ according to Schmidt decomposition. Firstly, we define a distinct measure of entanglement of two-qubit pure state as follows:

Definition 2.1. Let x be the minimal Schmidt coefficients of $\mathrm{tr}_A(|\psi\rangle\langle\psi|)$. We define entanglement of two-qubit pure state $|\psi\rangle$ as:

$$E_k(\psi) = \frac{x}{k} \wedge 1, \quad \forall k \in (0, 1], \tag{2}$$

where $\wedge$ is the min operator. Especially, $E_0(\psi) = \lim_{k\to 0} E_k(\psi) = 1$.

Remark 1. It is easy to see that $E_k(\psi)$ mentioned above is indeed entanglement monotones.

Similar to Ref.[11], let $C(\psi) = |\langle\psi|\sigma_2 \otimes \sigma_2\psi^*\rangle|$ denote the concurrence of a two-qubit pure state $|\psi\rangle$. Obviously, $C(\psi) = 2\sqrt{(1 - E_1(\psi))E_1(\psi)}$, where $|\psi^*\rangle$ is the complex conjugate $|\psi\rangle$ and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Now we discuss the entanglement monotones of two-qubit mixed states. Inspired by the ideas presented in Ref. [12], we define entanglement monotones of mixed state as follows:

Definition 2.2. The entanglement monotones of the mixed state ρ is defined as the average entanglement of the pure states of the decomposition, minimized over all decompositions of ρ:

$$E_k(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum p_i E_k(\psi_i), \quad \forall\, k \in (0, 1]. \tag{3}$$

Definition 2.3. (See [11].) The concurrence of two-qubit mixed state ρ is defined as

$$C(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum p_i C(\psi_i). \tag{4}$$

Following the Wootters' results [11], we can obtain that $C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where the λ_is are the square roots of the eigenvalues of $\rho(\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2)$ in decreasing order. Furthermore, for any mixed state ρ there always exists a pure-state ensemble with at most four states such that the following equation holds:

$$C(\rho) = \sum_{i=1}^{\leqslant 4} p_i C(\psi_i). \tag{5}$$

By Eq. (5), we have

Proposition 2.4.

$$E_k(\rho) = \frac{1 - \sqrt{1 - C^2(\rho)}}{2k} \wedge 1. \tag{6}$$

Proof. Note that there exists an ensemble $\{p_i; |\psi_i\}$ such that $C(\rho) = \sum_{i=1}^{\leqslant 4} p_i C(\psi_i)$. So $\frac{1-\sqrt{1-C^2(\rho)}}{2k} \wedge 1 = \frac{1-\sqrt{1-(\sum_{i=1}^{\leqslant 4} p_i C(\psi_i))^2}}{2k} \wedge 1 \leqslant \sum_{i=1}^{\leqslant 4} p_i \frac{1-\sqrt{1-C^2(\psi_i)}}{2k} \wedge 1 = \sum_i^{\leqslant 4} p_i E(\psi_i)$, where we use the convex of function $f(x) = \frac{1-\sqrt{1-x^2}}{2k} \wedge 1$ in the second inequality. This means that $\frac{1-\sqrt{1-C^2(\rho)}}{2k} \wedge 1$ is a lower bound of $\sum_i p_i E(\psi_i)$. Hence, we get $E(\rho) \leqslant \frac{1-\sqrt{1-C^2(\rho)}}{2k} \wedge 1$.

On the other hand, $\sum_i p_i C(\psi_i) = 2k \sum_i p_i \sqrt{(1 - E(\psi_i))E(\psi_i)} \wedge 1$. Define a function $g(x) = 2k\sqrt{(1-x)x} \wedge 1$. It is easy to see that $g(x)$ is concave and monotonically increasing on $[0, \frac{1}{2}]$. We therefore have $\sum p_i C(\psi_i) \leqslant 2k\sqrt{(1 - \sum_i p_i E(\psi_i))(\sum_i p_i E(\psi_i))} \wedge 1$. Since g is monotonically increasing, $\min_{\{p_i, |\psi_i\rangle\}} \sum p_i C(\psi_i) \leqslant f(\min_{\{p_i, |\psi_i\rangle\}} \sum p_i E(\psi_i)) = f(E(\rho))$ holds. Hence $E(\rho) \geqslant \frac{1-\sqrt{1-C^2(\rho)}}{2k} \wedge 1$. □

Remark 2. An ensemble that achieves the minimum in Eq. (4) is referred to an optimal one in this Letter.

Next we consider the entanglement transformation between two-qubit mixed states by LOCC. Let $\{p_i, |\psi_i\rangle\}$ and $\{q_j; |\phi_j\rangle\}$ be two optimal ensembles of two-qubit mixed states ρ and σ, respectively. We say

Definition 2.5. $\rho \xrightarrow{\text{LOCC}} \sigma$ if $\{p_i; |\psi_i\rangle\} \xrightarrow{\text{LOCC}} \{q_j; |\phi_j\rangle\}$.

Returning to the problem of entanglement transformation between two-qubit mixed states ρ and σ by LOCC, we have

Theorem 2.6.

$$\rho \xrightarrow{\text{LOCC}} \sigma \quad \textit{if and only if} \quad E_k(\rho) \geqslant E_k(\sigma), \quad \forall k \in (0, 1]. \tag{7}$$

Proof. It is easy to testify that the right hand of Eq. (3) satisfies the conditions of entanglement monotones. Since LOCC cannot