Physics Letters A 380 (2016) 1395–1400

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Quantum speed limit for mixed states using an experimentally realizable metric

Debasis Mondal *, Arun Kumar Pati *

Quantum Information and Computation Group, Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad, India

A R T I C L E I N F O

Article history:
Received 24 December 2015
Received in revised form 12 February 2016
Accepted 15 February 2016
Available online 18 February 2016
Communicated by P.R. Holland

Keywords:
Visibility
Quantum speed limit
CPTP maps
Quantum metrology
Margolus–Levitin bound
Mandelstam–Tamm bound

A B S T R A C T

Here, we introduce a new metric for non-degenerate density operator evolving along unitary orbit and show that this is experimentally realizable operation dependent metric on the quantum state space. Using this metric, we obtain the geometric uncertainty relation that leads to a new quantum speed limit (QSL). We also obtain a Margolus–Levitin bound and an improved Chau bound for mixed states. We propose how to measure this new distance and speed limit in quantum interferometry. Finally, we also generalize the QSL for completely positive trace preserving evolutions.

1. Introduction

In recent years, various attempts are being made in the laboratory to implement quantum gates, which are basic building blocks of a quantum computer. Performance of a quantum computer is determined by how fast one can apply these logic gates so as to drive the initial state to a final state. Then, the natural question that arises is: can a quantum state evolve arbitrarily fast? It turns out that quantum mechanics limits the evolution speed of any quantum system. In quantum information, study of these limits has found several applications over the years. Some of these include, but are not limited, to quantum metrology, quantum chemical dynamics, quantum control and quantum computation.

Extensive amount of work has already been done on the subject to "minimum time required to reach a target state" since the appearance of first major result by Mandelstam and Tamm [1]. However, the notion of quantum speed or speed of transportation of quantum state was first introduced by Anandan–Aharonov using the Fubini–Study metric [2] and subsequently, the same notion was defined in Ref. [3] using the Riemannian metric [4]. It was found that the speed of a quantum state on the projective Hilbert space is proportional to the fluctuation in the Hamiltonian of the system. Using the concept of Fubini–Study metric on the projective Hilbert space, a geometric meaning is given to the probabilities of a two-state system [5]. Furthermore, it was shown that the quantum speed is directly related to the super current in the Josephson junction [6]. In the last two decades, there have been various attempts made in understanding the geometric aspects of quantum evolution for pure as well for mixed states [7–56]. The quantum speed limit (QSL) for the driven [53] and the non-Markovian [52] quantum systems is introduced using the notion of Bures metric [61]. Very recently, QSL for physical processes was defined by Taddei et al. in Ref. [48] using the Bures metric and in the case of open quantum system the same is introduced by Campo et al. in Ref. [49] using the notion of relative purity [47]. In an interesting twist, it has been shown that QSL for multipartite system is bounded by the generalized geometric measure of entanglement [50].

It is worthwhile to mention that very recently, an experiment was reported [57], which is the only experiment performed, where only a consequence of the QSL had been tested and any experimental test of the speed limit itself is still lacking. In this paper, we introduce a new operation dependent metric, which can be measured experimentally in the interference of mixed states. We show that using this metric, it is possible to define a new lower limit for the evolution time of any system described by mixed state undergoing unitary evolution. We derive the QSL using the geometric uncertainty relation based on this new metric. We also obtain a Margolus–Levitin (ML) bound and an improved Chau bound for mixed states using our approach. We show that this bound for the

* Corresponding authors.
E-mail addresses: debamondal@hri.res.in (D. Mondal), akpati@hri.res.in (A.K. Pati).

http://dx.doi.org/10.1016/j.physleta.2016.02.018

evolution time of a quantum system is tighter than any other existing bounds for unitary evolutions. Most importantly, we propose an experiment to measure this new distance in the interference of mixed states. We argue that the visibility in quantum interference is a direct measure of distance for mixed quantum states. Finally, we generalize the speed limit for the case of completely positive trace preserving evolutions and get a new lower bound for the evolution time using this metric.

The organization of the paper is as follows. In section 2, we define the metric for the density operator along unitary path. Then, we use this metric to obtain new and tighter time bounds for unitary evolutions in section 3, followed by examples in section 4. In section 5, we show that bounds are experimentally measurable. Section 6 is for generalization of the metric and the time bounds for completely positive trace preserving (CPTP) maps followed by an example. Then, we conclude in section 7.

2. Metric along unitary orbit

Let $\mathcal{H}$ denotes a finite-dimensional Hilbert space and $\mathcal{L}(\mathcal{H})$ is the set of linear operators on $\mathcal{H}$. A density operator ρ is a Hermitian, positive and trace class operator that satisfies $\rho \geq 0$ and $\mathrm{Tr}(\rho) = 1$. Let ρ be a non-degenerate density operator with spectral decomposition $\rho = \sum_k \lambda_k |k\rangle\langle k|$, where λ_k's are the eigenvalues and $\{|k\rangle\}$'s are the eigenstates. We consider a system at time t_1 in a state ρ_1. It evolves under a unitary evolution and at time t_2, the state becomes $\rho_2 = U(t_2, t_1)\rho_1 U^\dagger(t_2, t_1)$. Any two density operators that are connected by a unitary transformation will give a unitary orbit. If $U(N)$ denotes the set of $N \times N$ unitary matrices on $\mathcal{H}^N$, then for a given density operator ρ, the unitary orbit is defined by $\rho' = \{U\rho U^\dagger : U \in U(N)\}$. The most important notion that has resulted from the study of interference of mixed quantum states is the concept of the relative phase between ρ_1 and ρ_2 and the notion of visibility in the interference pattern. The relative phase is defined by [58]

$$\Phi(t_2, t_1) = \mathrm{Arg}\mathrm{Tr}[\rho_1 U(t_2, t_1)] \tag{1}$$

and the visibility is defined by

$$V = |\mathrm{Tr}[\rho_1 U(t_2, t_1)]|. \tag{2}$$

Note that if $\rho_1 = |\psi_1\rangle\langle\psi_1|$ is a pure state and $|\psi_1\rangle = |\psi(t_1)\rangle \to |\psi_2\rangle = |\psi(t_2)\rangle = U(t_2, t_1)|\psi(t_1)\rangle$, then $|\mathrm{Tr}(\rho_1 U(t_2, t_1))|^2 = |\langle\psi(t_1)|\psi(t_2)\rangle|^2$, which is nothing but the fidelity between two pure states. The quantity $\mathrm{Tr}[\rho_1 U(t_2, t_1)]$ represents the probability amplitude between ρ_1 and ρ_2, which are unitarily connected. Therefore, for the unitary orbit $|\mathrm{Tr}(\rho_1 U(t_2, t_1))|^2$ represents the transition probability between ρ_1 and ρ_2.

All the existing metrics on the quantum state space give rise to the distance between two states independent of the operation. Here, we define a new distance between two unitarily connected states of a quantum system. This distance not only depends on the states but also depends on the operation under which the evolution occurs. Whether a state of a system will evolve to another state depends on the Hamiltonian which in turn fixes the unitary orbit. Let the mixed state traces out an open unitary curve $\Gamma : t \in [t_1, t_2] \to \rho(t)$ in the space of density operators with "end points" ρ_1 and ρ_2. If the unitary orbit connects the state ρ_1 at time t_1 to ρ_2 at time t_2, then the (pseudo-)distance between them is defined by

$$D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2})^2 := 4(1 - |\mathrm{Tr}[\rho_1 U(t_2, t_1)]|^2), \tag{3}$$

which also depends on the orbit, i.e., $U(t_2, t_1)$. We will show that it is indeed a metric, i.e., it satisfies all the axioms to be a metric.

We know that for any operator A and a unitary operator U, $|\mathrm{Tr}(AU)| \leq \mathrm{Tr}|A|$ with equality for $U = V^\dagger$, where $A = |A|V$ is the polar decomposition of A [59]. Considering $A = \rho = |\rho|$, we get $|\mathrm{Tr}[\rho_1 U(t_2, t_1)]| \leq 1$. This proves the non-negativity, or separation axiom. It can also be shown that $D_U(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 0$ if and only if there is no evolution along the unitary orbit, i.e., $\rho_1 = \rho_2$ and $U = I$. If there is no evolution along the unitary orbit, then we have $U(t_2, t_1) = I$, i.e., trivial or global cyclic evolution, i.e., $\rho_2 = U(t_2, t_1)\rho_1 U^\dagger(t_2, t_1) = \rho_1$, which in turn implies $D_U(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 0$. To see the converse, i.e., if $D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 0$, then we have no evolution, consider the purification. We have $D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 4(1 - |\langle\Psi_{AB}(t_1)|\Psi_{AB}(t_2)\rangle|^2)$ where $|\Psi_{AB}(t_2)\rangle = U_A(t_2, t_1) \otimes I_B|\Psi_{AB}(t_1)\rangle$ such that $\mathrm{Tr}_B(|\Psi_{AB}(t_1)\rangle\langle\Psi_{AB}(t_1)|) = \rho_1$ and $\mathrm{Tr}_B(|\Psi_{AB}(t_2)\rangle\langle\Psi_{AB}(t_2)|) = \rho_2$. In the extended Hilbert space, $D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 0$ implies $|\langle\Psi_{AB}(t_1)|\Psi_{AB}(t_2)\rangle|^2 = 1$ and hence, $\Psi_{AB}(t_1)$ and $\Psi_{AB}(t_2)$ are same up to $U(1)$ phases. Therefore, in the extended Hilbert space, $D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2}) = 0$ if and only if there is no evolution. But in the original Hilbert space there are non-trivial cyclic evolutions for which $D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2}) \neq 0$ in spite of the fact that $\rho_1 = \rho_2$. To prove the symmetry axiom, we show that the quantity $|\mathrm{Tr}[\rho_1 U(t_2, t_1)]|$ is symmetric with respect to the initial and the final states. In particular, we have

$$|\mathrm{Tr}[\rho_1 U(t_2, t_1)]| = |\mathrm{Tr}[\rho_2 U(t_1, t_2)]| = |\mathrm{Tr}[\rho_2 U(t_2, t_1)]|. \tag{4}$$

To see that the new distance satisfies the triangle inequality, consider its purificaton. Let $\rho_A(t_1)$ and $\rho_A(t_2)$ be two unitarily connected mixed states of a quantum system A. If we consider the purification of $\rho_A(t_1)$, then we have $\rho_A(t_1) = \mathrm{Tr}_B[|\Psi_{AB}(t_1)\rangle\langle\Psi_{AB}(t_1)|]$, where $|\Psi_{AB}(t_1)\rangle = (\sqrt{\rho_A(t_1)}V_A \otimes V_B)|\alpha\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, V_A, V_B are local unitary operators and $|\alpha\rangle = \sum_i |i^A i^B\rangle$. The evolution of $\rho_A(t_1)$ under $U_A(t_2, t_1)$ is equivalent to the evolution of the pure state $|\Psi_{AB}(t_1)\rangle$ under $U_A(t_2, t_1) \otimes I_B$ in the extended Hilbert space. Thus, in the extended Hilbert space, we have $|\Psi_{AB}(t_1)\rangle \to |\Psi_{AB}(t_2)\rangle = U_A(t_2, t_1) \otimes I_B|\Psi_{AB}(t_1)\rangle$. So, the transition amplitude between two states is given by $\langle\Psi_{AB}(t_1)|\Psi_{AB}(t_2)\rangle = \mathrm{Tr}[\rho_A(t_1)U_A(t_2, t_1)]$. This simply says that the expectation value of a unitary operator $U_A(t_2, t_1)$ in a mixed state is equivalent to the inner product between two pure states in the enlarged Hilbert space. Since, in the extended Hilbert space the purified version of the metric satisfies the triangle inequality, hence the triangle inequality holds also for the mixed states. Thus, $D_{U(t_2,t_1)}(\Gamma_{\rho_1}, \Gamma_{\rho_2})$ is a distance in the extended Hilbert space and a pseudo-distance in the original Hilbert space. If ρ_1 and ρ_2 are two pure states, which are unitarily connected then our new metric is the Fubini-Study metric [3,2,60] on the projective Hilbert space $\mathbf{CP}(\mathcal{H})$.

Now, imagine that two density operators differ from each other in time by an infinitesimal amount, i.e., $\rho(t_1) = \rho(t) = \sum_k \lambda_k |k\rangle\langle k|$ and $\rho(t_2) = \rho(t + dt) = U(dt)\rho(t)U^\dagger(dt)$. Then, the infinitesimal distance between them is given by

$$dD^2_{U(dt)}(\Gamma_{\rho(t_1)}, \Gamma_{\rho(t_2)}) = 4(1 - |\mathrm{Tr}[\rho(t)U(dt)]|^2). \tag{5}$$

If we use the time independent Hamiltonian H for the unitary operator, then keeping terms upto second order, the infinitesimal distance (we drop the subscript) becomes

$$\begin{aligned} dD^2 &= \frac{4}{\hbar^2}[\mathrm{Tr}(\rho(t)H^2) - [\mathrm{Tr}(\rho(t)H)]^2]dt^2 \\ &= \frac{4}{\hbar^2}[\sum_k \lambda_k \langle k|H^2|k\rangle - (\sum_k \lambda_k \langle k|H|k\rangle)^2]dt^2 \\ &= \frac{4}{\hbar^2}[\sum_k \lambda_k \langle\dot{k}|\dot{k}\rangle - (i\sum_k \lambda_k \langle k|\dot{k}\rangle)^2]dt^2, \end{aligned} \tag{6}$$

where in the last line we used the fact that $i\hbar|\dot{k}\rangle = H|k\rangle$. Therefore, the total distance traveled during an evolution along the unitary orbit is given by