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Comprehensive theory for reduction of products of spin operators

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ABSTRACT

We present a comprehensive theory that reduces the total power of products of spin operators. This theory improves the previous one [P.J. Jensen, F. Aguilera-Granja, Phys. Lett. A 269 (2000) 158] in two aspects. One is that for the set of spin operators S^+, S^-, S^z, a new method is suggested where the expansion coefficients in the reduction formula can be solved from linear equations. This new method is of direct physical meaning and is easier to handle. The other is that we show a method to reduce the products of another set of spin operators S^x, S^y, S^z. For this set of operators, the use of permutation regulation of $x \to y$, $y \to z$ and $z \to x$ can save much time in obtaining some reduction formula. The present comprehensive theory enables one to deal more easily with the decoupling problems in Green' function theory where the set of either S^+, S^-, S^z or S^x, S^y, S^z operators is used.

1. Introduction

Usually, the systems consisted of local magnetic moments with exchange interactions between them are modeled by Heisenberg Hamiltonian. To describe low-dimensional magnetic systems with spontaneous magnetization, or a long-range ordering at finite temperature, anisotropy should be considered. The single-ion anisotropy is mostly used [1–15]. In fact, this kind of anisotropy was introduced long before [16,17] in three-dimensional systems. The anisotropy gives rise to one or more easy axes in crystal lattices so that plays important roles on the behavior of magnetization. For instances, the spin reorientation occurs when an external magnetic field is applied, and the orientation of the spins is determined by the directions of the easy axes and the field [18]. When the field varies, hysteresis loops will appear due to the anisotropy [10–13,18]. Besides, this kind of anisotropy is one of the factors that cause the formation of magnetic domains in the crystals. There may be single-ion anisotropy terms with two, four and even higher orders [1,8].

A well-known method to deal with the Heisenberg model is the method of equation of motion of the many-body Green's functions [19,20]. This method requires some approximations in decoupling higher order Green's functions in order to break off the equation chain. For those functions concerning spins on different crystal sites, often random phase approximation is used [15–18]. While for those concerning spins on the same sites, henceforth referred to as on-site terms, Anderson–Callen decoupling approximation is used [2,16]. The single-ion anisotropy term is a typical example of the on-site term. It should be noticed that even the anisotropy is absent, one may also has to take into account the cases of on-site terms as long as the decoupling of higher order Green's functions are considered [14,21–25]. This is because the Green's functions will contain the products of at least three spin operators belonging to sites, say, A, B and C, respectively. Suppose that site A is the nearest neighbor (nn) of B and B in turn is the nn of sit C. If the summation of A sites covers all the nn sites of B, then one of A sites is in fact just the site C. Thus the product of two on-site spins is encountered. One hopes to treat the on-site terms exactly, but the cases have been very few [1].

Since the spirit of decoupling a higher order Green's function is to express it in terms of lower order Green's functions, the best way to treat a product of on-site spin operators is to reduce the power of the product. The power of an operator written as a product of several component operators is defined as the sum of the exponents of the components. For instance, the power of $A^l B^m C^n$ is $l+m+n$. The reduction of the power is possible because of a fundamental property of the raising and lowing operators $S^{\pm}$:

$$(S^{\pm})^{2S+1} = 0. \tag{1}$$

The reduction of a product of spin operators $(S^+)^l (S^-)^m (S^z)^n$, $l+m+n \geqslant 2S+1$ means that it can be expanded by terms with

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powers less than $2S+1$. In this Letter, we use S to denote the (integer or half-integer) spin quantum number and m and n to denote natural numbers with the condition $m, n \leqslant 2S+1$. Jensen and Aguilera-Granja [26] first gave a method to reduce the total power by means of the commutator relations between spin operators. Their main results were the reduction formula of $(S^z)^{2S+1-m}(S^+)^m$ and its complex conjugate.

We find that the reduction method is not unique, and that it deserves to exploit an easier way to obtain a comprehensive formula of the reduction. Our first aim in this Letter is to present the formula.

Since the spin operators S^+, S^-, S^z were originated from S^x, S^y, S^z, the two sets are equivalent to each other. The original spin operators S^x, S^y, S^z are also used from time to time in dealing with Heisenberg systems by means of the Green's function method [13,15,27]. Therefore, a reduction method about the products $(S^x)^l(S^y)^m(S^z)^n$, $l+m+n \geqslant 2S+1$ is also desirable. Our second aim in this Letter is to give the method.

This Letter is arranged as follows. First of all, we prove in Section 2 the simplest reductions:

$$S^z(S^\pm)^{2S} = \pm S(S^\pm)^{2S} \tag{2a}$$

and

$$(S^\pm)^{2S}S^z = \mp S(S^\pm)^{2S}. \tag{2b}$$

In doing so, we give our main idea that instead of manipulating spin operators as in Ref. [26], we prefer to resort to the action of the operators on the eigen states of operator S^z. We stress that Eqs. (1) and (2) are valid in the sense that when both sides of the equations act on any state, the equations are always correct. Thus, the physical meaning of the identities is apparent. Then with this idea, we show in Section 3 that in the reduction formula of products $(S^z)^{2S+1-m}(S^\pm)^m$, the expansion coefficients can be solved by linear equations. This new method is easier to understand and to handle. In Section 4, we present a routine how to reduce the products of S^x, S^y, S^z. Finally, in Section 5 we present a summary. Through out this paper $\hbar = 1$ is used.

2. Proof of Eqs. (2)

It is well known that there are $2S+1$ eigen states, denoted as $|i\rangle$, of operator S^z:

$$S^z|i\rangle = i|i\rangle, \quad i = -S, -S+1, \ldots, S. \tag{3}$$

The action effect of $S^\pm$ on the states are

$$S^\pm|i\rangle = u^\pm_{S,i}|i\pm 1\rangle, \tag{4}$$

where $u^\pm_{S,i} = \sqrt{(S\mp i)(S\pm i+1)}$ which, however, will not be used below. We emphasize that

$$S^\pm|\pm S\rangle = 0. \tag{5}$$

An arbitrary state, denoted as $|w\rangle$, can be put down as a linear combination of the eigenstates of S^z:

$$|w\rangle = \sum_{i=-S}^{S} w_i|i\rangle. \tag{6}$$

We point out the fact that Eq. (1) is valid because when the operators $(S^\pm)^{2S+1}$ act on $|w\rangle$, the result is always zero. Now consider the operator $S^z(S^+)^{2S}$ and let it act on $|w\rangle$. Because of Eqs. (4) and (5), the effect of action of $(S^+)^{2S}$ on $|i\rangle$, $i = -S+1, -S+2, \ldots, S$ is zero with only one exception where $i = -S$. Hence, $(S^+)^{2S}|w\rangle = (S^+)^{2S}w_{-S}|{-S}\rangle$. The resultant state of $(S^+)^{2S}|{-S}\rangle$ is of cause state $|S\rangle$. Then the action of operator S^z on the resulted state is $S^z|S\rangle = S|S\rangle$. Therefore, the effects of actions of $S^z(S^+)^{2S}$ and $S(S^+)^{2S}$ on $|w\rangle$ are the same. We can put down

$$S^z(S^+)^{2S}|w\rangle = S(S^+)^{2S}|w\rangle. \tag{7}$$

Since $|w\rangle$ is an arbitrary state, we obtain $S^z(S^+)^{2S} = S(S^+)^{2S}$. In the same manner, one can prove $S^z(S^-)^{2S} = -S(S^-)^{2S}$. By taking complex conjugate, one obtains $(S^-)^{2S}S^z = S(S^-)^{2S}$ and $(S^+)^{2S}S^z = -S(S^+)^{2S}$, respectively. Thus we have proved Eq. (2).

In the same way, one can show that

$$\left(S^z\right)^n(S^\pm)^{2S} = (\pm S)^n(S^\pm)^{2S} \tag{8a}$$

and their complex conjugate

$$(S^\pm)^{2S}\left(S^z\right)^n = (\mp S)^n(S^\pm)^{2S}. \tag{8b}$$

Eq. (2) are just the special cases of $n=1$ of Eqs. (8).

It is also easily shown that the combination of Eqs. (3) and (4) yields a useful formula:

$$\left(S^z\right)^n(S^\pm)^m|i\rangle = (i\pm m)^n(S^\pm)^m|i\rangle. \tag{9}$$

By use of Eqs. (9), one can even go further to put down a generalized formula as follows:

$$(S^\pm)^{2S-m}\left(S^z\right)^n(S^\pm)^m = (\mp 1)^n(S-m)^n(S^\pm)^{2S}. \tag{10}$$

When taking $m = 2S$ or 0, one obtains Eq. (8a) or (8b), respectively. Eqs. (9) will be utilized in the next subsection.

3. Reduction of $(S^z)^m(S^\pm)^{2S+1-m}$

We now consider the reduction of operators $(S^z)^m(S^+)^{2S+1-m}$ with their power being $2S+1$. After the reduction, the operator should be expressed by a linear combination of operator products with the powers not larger than $2S$. When $m=1$, the formula is Eq. (2). The idea of reduction is still to inspect the action of the operator on an arbitrary state. We know that the operator $(S^z)^m$ does not alter the state $|i\rangle$ while $(S^+)^{2S+1-m}$ raises it to $|i+2S+1-m\rangle$. The operators reduced from $(S^z)^m(S^+)^{2S+1-m}$ should have the same effect as $(S^z)^m(S^+)^{2S+1-m}$ does. It is therefore reasonable assuming that the factor $(S^+)^{2S+1-m}$ remains unchanged after the reduction. Consequently, the reduction should have the following form:

$$\left(S^z\right)^m(S^+)^{2S+1-m} = \sum_{n=0}^{m-1} a_n^{(S,m)}\left(S^z\right)^n(S^+)^{2S+1-m}, \tag{11}$$

where $a_n^{(S,m)}$'s are expansion coefficients. This reduction formula is in fact unique. Our aim is to give a method to determine the coefficients. To this aim, we let the operators on both sides of Eq. (11) act on the state of Eq. (6) as following:

$$\begin{aligned}&\left(S^z\right)^m(S^+)^{2S+1-m}\sum_{i=-S}^{S} w_i|i\rangle\\ &\quad= \sum_{n=0}^{m-1} a_n^{(S,m)}\left(S^z\right)^n(S^+)^{2S+1-m}\sum_{i=-S}^{S} w_i|i\rangle.\end{aligned} \tag{12}$$

Substituting Eq. (9) into Eq. (12) results in

$$\begin{aligned}&\sum_{i=-S}^{S} w_i(i+2S+1-m)^m(S^+)^{2S+1-m}|i\rangle\\ &\quad= \sum_{n=0}^{m-1} a_n^{(S,m)}\sum_{i=-S}^{S} w_i(i+2S+1-m)^n(S^+)^{2S+1-m}|i\rangle.\end{aligned} \tag{13}$$

It is obvious that when $i \geqslant -S+m$, $(S^+)^{2S+1-m}|i\rangle = 0$ because of Eq. (5). Eq. (13) can be rearranged as