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Note on integrability of certain homogeneous Hamiltonian systems

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ABSTRACT

In this paper we investigate a class of natural Hamiltonian systems with two degrees of freedom. The kinetic energy depends on coordinates but the system is homogeneous. Thanks to this property it admits, in a general case, a particular solution. Using this solution we derive necessary conditions for the integrability of such systems investigating differential Galois group of variational equations.

1. Introduction

It seems that the most effective methods of proving non-integrability are based on application of the differential Galois theory. For Hamiltonian systems necessary conditions for the integrability in the Liouville sense are given by the Morales–Ramis theorem.

Theorem 1.1 *(Morales-Ruiz and Ramis). Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of a phase curve $\mathbf{\Gamma}$ corresponding to a particular solution. Then, the identity component $\mathcal{G}^0$ of the differential Galois group $\mathcal{G}$ of variational equations along $\mathbf{\Gamma}$ is Abelian.*

For a detailed exposition and a proof see e.g. [4,5].

The above theorem has found a very effective application for natural systems given by the following Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{n}p_i^2+V(\boldsymbol{q}), \tag{1.1}$$

where $V(\boldsymbol{q})$ is a homogeneous function of degree $k\in\mathbb{Z}$, and $\boldsymbol{q}=(q_1,\dots,q_n)$ and $\boldsymbol{p}=(p_1,\dots,p_n)$ are the generalised coordinates and momenta, respectively. Let us note that for application of Theorem 1.1 we have to know a particular solution of the considered system. In general it is a difficult problem how to find such a solution. However for systems given by (1.1) with a homogeneous potential $V(\boldsymbol{q})$ it is well known that if $\boldsymbol{d}\in\mathbb{C}^n$ is a non-zero solution of nonlinear system $V'(\boldsymbol{d})=\boldsymbol{d}$, then functions

$$\boldsymbol{q}(t)=\varphi(t)\boldsymbol{d},\qquad \boldsymbol{p}(t)=\varphi(t)\boldsymbol{d},\qquad \ddot{\varphi}=-\varphi^{k-1}, \tag{1.2}$$

determine a particular solution of Hamilton's equations. The variational equations along this solution split into a direct product of second order equations of the form

$$\ddot{x}=-\lambda\varphi(t)^{k-2}x, \tag{1.3}$$

where λ is an eigenvalue of Hessian $V''(\boldsymbol{d})$. The necessary conditions for the integrability have the form of arithmetic restrictions on λ, see e.g. [4,5]. The crucial role in derivation of these conditions plays the Yoshida change of independent variable which transforms equation (1.3) into the Gauss hypergeometric equation [8].

Hamiltonian (1.1) describes a particle moving under influence of potential forces in flat Euclidean space $\mathbb{R}^n$. It is a natural to ask what is an analog of homogeneous systems in curved spaces. There is no obvious answer to this question. We have to take into account the form of metric of the configuration space as well as the form of the potential. We leave a general discussion of this problem to a separate paper and here we consider systems with two degrees of freedom given by the following Hamiltonian

$$H=T+V,\quad T=\frac{1}{2}r^{m-k}\left(p_r^2+\frac{p_\varphi^2}{r^2}\right),\quad V=r^mU(\varphi), \tag{1.4}$$

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Table 1
Integrability table. Here $k, m, p, q \in \mathbb{Z}$ and $k \neq 0$.

No.	k	m	$\mathcal{I}(k,m)$
1	$k=-2(mp+1)$	m	$\mathbb{C}$
2	$k \in \mathbb{Z}\setminus\{0\}$	m	$\mathcal{I}_a(k,m)$
3	$k=2(mp-1)\pm\frac{1}{3}m$	$3q$	$\bigcup_{i=0}^{6}\mathcal{I}_i(k,m)$
4	$k=2(mp-1)\pm\frac{1}{2}m$	$2q$	$\mathcal{I}_a(k,m)\cup\mathcal{I}_4(k,m)$
5	$k=2(mp-1)\pm\frac{3}{5}m$	$5q$	$\mathcal{I}_a(k,m)\cup\mathcal{I}_3(k,m)\cup\mathcal{I}_6(k,m)$
6	$k=2(mp-1)\pm\frac{1}{5}m$	$5q$	$\mathcal{I}_a(k,m)\cup\mathcal{I}_3(k,m)\cup\mathcal{I}_5(k,m)$

where m and k are integers, and $k \neq 0$. If we consider (r,φ) as the polar coordinates, then the kinetic energy corresponds to a singular metric on a plane or a sphere. We assume that $U(\varphi)$ is a complex meromorphic function of variable $\varphi \in \mathbb{C}$, and we do not require that $U(\varphi)$ is periodic.

The main result of this paper is the following theorem which gives necessary conditions for the integrability of Hamiltonian systems given by (1.4). For its formulation we need to define the following sets

$$\mathcal{I}_0(k,m) := \left\{ \frac{1}{k}(mp+1)(2mp+k) \;\middle|\; p \in \mathbb{Z} \right\}, \tag{1.5}$$

$$\mathcal{I}_1(k,m) := \left\{ \frac{1}{2k}(mp-2)(mp-k) \;\middle|\; p = 2r+1, r \in \mathbb{Z} \right\}, \tag{1.6}$$

$$\mathcal{I}_2(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{2}\right)^2-(k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\}, \tag{1.7}$$

$$\mathcal{I}_3(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{3}\right)^2-(k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\}, \tag{1.8}$$

$$\mathcal{I}_4(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{4}\right)^2-(k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\}, \tag{1.9}$$

$$\mathcal{I}_5(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{5}\right)^2-(k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\}, \tag{1.10}$$

$$\mathcal{I}_6(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{2}{5}\right)^2-(k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\}, \tag{1.11}$$

and we put

$$\mathcal{I}_a(k,m) := \mathcal{I}_0(k,m)\cup\mathcal{I}_1(k,m)\cup\mathcal{I}_2(k,m). \tag{1.12}$$

Theorem 1.2. *Assume that $U(\varphi)$ is a complex meromorphic function and there exists $\varphi_0 \in \mathbb{C}$ such that $U'(\varphi_0)=0$ and $U(\varphi_0)\neq 0$. If the Hamiltonian system defined by Hamiltonian (1.4) is integrable in the Liouville sense, then number*

$$\lambda := 1 + \frac{U''(\varphi_0)}{kU(\varphi_0)}, \tag{1.13}$$

belongs to set $\mathcal{I}(k,m)$ which is defined by Table 1.

The above theorem tells us that if $k=-2(mp+1)$, then the Morales–Ramis Theorem 1.1 does not give any obstruction for the integrability of the considered systems. Let us notice that this is an infinite family of systems. For systems (1.1) with homogeneous potentials only two cases of this type are such distinguished, namely $k=\pm 2$ [4,5].

For each pair (k,m) of integers which do not satisfy relation $k=-2(pm+1)$, $p\in\mathbb{Z}$, Theorem 1.2 restricts admissible values λ to the set $\mathcal{I}_a(k,m)$. If m is not a multiple of 2, 3, and 5 these are the only restrictions. Otherwise, if m is a multiple of $q\in\{2,3,5\}$, and k takes appropriate value, then the set of admissible values of λ contains additional elements. These are Cases 3–6 in Table 1.

Let us note that the above theorem remains valid for rational k and m. In such extended version we require that k is a non-zero rational number, and the restriction contained in the third column of Table 1 can be ignored. For the proof of this extended version one has to apply a reasoning similar to that one used in [1].

Let us remark that there is also other possibility to generalise systems given by (1.1) with homogeneous potentials in such a way that they will admit a straight line particular solution and the variational equations can be reduced to a direct product of hypergeometric equations. In [6] the authors consider system with Hamiltonian

$$H = T(\boldsymbol{p}) + V(\boldsymbol{q}), \tag{1.14}$$

where T and V are homogeneous functions of integer degrees. To find a straight line particular solution one must solve overdetermined system of nonlinear equations

$$T'(\boldsymbol{c}) = \boldsymbol{c}, \qquad V'(\boldsymbol{c}) = \boldsymbol{c},$$

that has a solution only in special cases. Moreover, this generalisation does not have a form of a natural Hamiltonian system. In other words, except the case when $\deg T = 2$, it cannot be considered as a Hamiltonian function of a point in a curved space.

2. Proof of Theorem 1.2

Equations of motion corresponding to Hamiltonian (1.4) have the form

$$\begin{aligned}
\dot r &= \frac{\partial H}{\partial p_r} = r^{m-k}p_r,\\
\dot p_r &= -\frac{\partial H}{\partial r}\\
&= r^{m-k-3}p_\varphi^2 - \frac{1}{2}(m-k)r^{m-k-1}\left(p_r^2+\frac{p_\varphi^2}{r^2}\right) - mr^{m-1}U(\varphi),\\
\dot\varphi &= \frac{\partial H}{\partial p_\varphi} = r^{m-k-2}p_\varphi,\\
\dot p_\varphi &= -\frac{\partial H}{\partial \varphi} = -r^m U'(\varphi).
\end{aligned} \tag{2.1}$$

If $U'(\varphi_0)=0$ for a certain $\varphi_0\in\mathbb{C}$, then system (2.1) has two dimensional invariant manifold

$$\mathcal{N} = \left\{(r,p_r,\varphi,p_\varphi)\in\mathbb{C}^4 | \varphi=\varphi_0,\ p_\varphi=0\right\}. \tag{2.2}$$

Indeed, equations (2.1) restricted to $\mathcal{N}$ read

$$\begin{aligned}
\dot r &= r^{m-k}p_r,\\
\dot p_r &= -\frac{1}{2}(m-k)r^{m-k-1}p_r^2 - mr^{m-1}U(\varphi_0).
\end{aligned} \tag{2.3}$$

Hence, $\mathcal{N}$ is foliated by phase curves parametrised by energy E

$$E = \frac{1}{2}r^{m-k}p_r^2 + r^m U(\varphi_0). \tag{2.4}$$

Taking into account that $\dot r = r^{m-k}p_r$ we can rewrite equation (2.4) in the form

$$\dot r^2 = 2r^{m-k}\left\{E - r^m U(\varphi_0)\right\}. \tag{2.5}$$

Let $[R, P_R, \Phi, P_\Phi]^T$ denote the variations of $[r, p_r, \varphi, p_\varphi]^T$. Then, the variational equations along a particular solution lying on $\mathcal{N}$ take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} R \\ P_R \\ \Phi \\ P_\Phi \end{bmatrix} = \boldsymbol{C}\begin{bmatrix} R \\ P_R \\ \Phi \\ P_\Phi \end{bmatrix}, \tag{2.6}$$