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An exactly solvable system from quantum optics

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ABSTRACT

We investigate a generalisation of the Rabi system in the Bargmann–Fock representation. In this representation the eigenproblem of the considered quantum model is described by a system of two linear differential equations with one independent variable. The system has only one irregular singular point at infinity. We show how the quantisation of the model is related to asymptotic behaviour of solutions in a vicinity of this point. The explicit formulae for the spectrum and eigenfunctions of the model follow from an analysis of the Stokes phenomenon. An interpretation of the obtained results in terms of differential Galois group of the system is also given.

1. Introduction and results

In paper [1] we proposed a general method which allows to determine spectra of quantum systems given in the Bargmann representation. This representation is very useful for systems with a Hilbert space which is a product of a finite and infinite-dimensional spaces. A typical example is a Hamiltonian for spin degrees of freedom of some particles, characterised by n-dimensional σ matrices, coupled to a bosonic field via annihilation and creation operators a, and $a^\dagger$. Then n-component wave function $\psi = (\psi_1, \dots, \psi_n)$ is an element of Hilbert space $\mathcal{H}^n = \mathcal{H} \times \cdots \times \mathcal{H}$, where $\mathcal{H}$ is the Bargmann–Fock Hilbert space of entire functions. The scalar product in $\mathcal{H}$ is given by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} \overline{f(z)} g(z) \mathrm{e}^{-|z|^2}\,\mathrm{d}x\,\mathrm{d}y, \quad z = x + \mathrm{i}\,y.$$

Since operators a and $a^\dagger$ are represented by ∂_z and multiplication by z, respectively, for clearly $[\partial_z, z] = 1$, stationary Schrödinger equation $H\psi = E\psi$ becomes the system of linear ordinary differential equations. Usually such systems have a certain number of regular singular points in the complex z plane, and possibly an irregular point at infinity. Determination of the spectrum consists in finding such values of E that all the components of wave function ψ_i, with $i = 1, \dots, n$, are elements of the Hilbert space $\mathcal{H}$.

In paper [1] we have shown that a solution of the eigenvalue problem can be reduced to checking the following three conditions.

1. Local conditions. At each regular singular point $z = s$ there exists at least one solution which is holomorphic in an open set containing s.
2. Global conditions. At each singular point we can select a local holomorphic solution in such a way that they are analytic continuations of each other. That is, they are local representations of an entire solution.
3. Normalisation conditions. The entire solution selected above must have a finite Bargmann norm.

The first two conditions were analysed in detail in [1] and in [2] we gave their effective application for determination of full spectrum of the Rabi system.

The third condition is highly non-trivial. This is so because "most" of entire functions do not belong to $\mathcal{H}$. In order to analyse this condition we need to characterise the growth of an entire solution $f(z)$ of a system of linear differential equations. In a neighbourhood of a regular singular point the growth of $f(z)$ is polynomial. Hence, if the considered system has only regular singular points, and $f(z)$ is its entire solution, then $f(z)$ has a finite Bargmann norm. Normalisation conditions can give non-trivial constrains on entire solution only if the linear system has an irregular singularity, for example at infinity.

The growth of entire function $f(z)$ is described by means of the following function:

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$$M(r) := \max_{|z|=r} |f(z)|. \tag{1}$$

It is used to define two numbers which characterise properties of the growth. The order, or the growth order ϱ of $f(z)$ is defined as the limit

$$\varrho := \lim_{r\to\infty} \sup \frac{\ln(\ln M(r))}{\ln r}, \quad \text{with} \quad 0 \le \varrho \le \infty. \tag{2}$$

For an entire function of finite order $\varrho < \infty$, its type σ is defined as

$$\sigma := \lim_{r\to\infty} \sup \frac{\ln M(r)}{r^{\varrho}}. \tag{3}$$

If $f(z)$ belongs to $\mathcal{H}$, then one can prove the following facts [3]:

1. $f(z)$ is of order $\varrho \le 2$.
2. If $\varrho = 2$, then $f(z)$ is of type $\sigma \le \frac{1}{2}$.

If $\varrho = 2$ and $\sigma = \frac{1}{2}$, then the question is whether $f(z) \in \mathcal{H}$ requires a separate investigation. For additional details see [4].

It is well known that in a neighbourhood of a regular singular point, a formal procedure allows to find formal series which satisfy the equation. It appears that this formal series is convergent, so it is a local solution of the equation. In a vicinity of irregular singular point a formal procedure gives formal expressions which satisfy the equation, however the series are generally divergent. It is known that these formal expressions, give the asymptotic expansion of a solution valid in certain sectors radially extending from the vertex localised at this singularity. Borders of sectors are determined by the so-called Stokes lines. The asymptotic expansions change when we pass from one sector to another. These changes are governed by the so-called Stokes matrices.

The normalisation condition of an entire solution $f(z)$ implies that at each sector S the following integral

$$\int_S |f(z)|^2 e^{-|z|^2}\, dx\, dy, \tag{4}$$

has a finite value. Thus, at each sector it has good asymptotics which guarantees the convergence of the above integral. This implies that although a solution $f(z)$ can change its asymptotic expansion from sector to sector, these changes are restricted only to good asymptotics. This property implies that all Stokes matrices have a common invariant subspace, in particular case just one common eigenvector.

The aim of this paper is to show that the above considerations can be applied effectively. We consider system given by the following Hamiltonian

$$H = \left(\omega + \frac{U}{2}\sigma_z\right) a^\dagger a + \frac{\omega_0}{2}\sigma_z + g\sigma_x(a^\dagger + a), \tag{5}$$

where σ_x, σ_z are the Pauli spin matrices and ω_0, ω, g and U are parameters. For $U = 0$, it coincides with the Hamiltonian of the Rabi model [5] describing interaction of a two-level atom with a single harmonic mode of the electromagnetic field. Hamiltonian (5) was proposed in [6,7]. The term $\frac{U}{2}\sigma_z a^\dagger a$ can be interpreted as a nonlinear coupling between the atom and the cavity.

In Bargmann–Fock representation, the stationary Schrödinger equation $H\psi = E\psi$, with $\psi = (\psi_1, \psi_2)$, have the form

$$\begin{aligned} &\left(\omega + \frac{U}{2}\right) z\psi_1' + \frac{\omega_0}{2}\psi_1 + g\psi_2' + gz\psi_2 = E\psi_1, \\ &\left(\omega - \frac{U}{2}\right) z\psi_2' - \frac{\omega_0}{2}\psi_2 + g\psi_1' + gz\psi_1 = E\psi_2. \end{aligned} \tag{6}$$

In our paper [1] we calculated the spectrum of H for the generic case, i.e., when the above system has two regular singular points. This requires that $U^2 \neq 4\omega^2$, and the quantisation of the energy spectrum results from the condition that system (6) admits an entire solution. In that case, all entire solutions of (6) have a finite norm.

In this paper we investigate the remaining cases for which $U^2 = 4\omega^2$. For $U = +2\omega$ system (6) can be rewritten in the form

$$\begin{aligned} \psi_1'(z) &= -z\psi_1(z) + \frac{2E+\omega_0}{2g}\psi_2(z), \\ \psi_2'(z) &= \frac{4\omega z^2 + 2E - \omega_0}{2g}\psi_1(z) \\ &\quad - \frac{z\left[g^2 + \omega(2E+\omega_0)\right]}{g^2}\psi_2(z). \end{aligned} \tag{7}$$

The system for the case of $U = -2\omega$ can be obtained from the above equations by a simple change $\omega_0 \to -\omega_0$, and the interchange ψ_1 with ψ_2. Hence, we consider only the case $U = +2\omega$.

System (7) has no singular points in the finite part of $\mathbb{C}$, so all its solutions are entire functions. Infinity is the only and irregular singular point. The system can have solutions which grow fast enough to make the Bargmann norm infinite. Thus, to determine the spectrum of the problem we have to find all values of E for which system admits a solution with a moderate growth at infinity, such that its Bargmann norm is finite.

We give a full answer to this question. That is we specify explicitly a countable number of energy values for which the corresponding eigenfunctions are also given explicitly and have a finite Bargmann norm.

The energy axis is divided into two disjoint intervals, each of them with its own countable family of eigenvalues. To be more precise, our main result is as follows. Let

$$x := 1 + \omega g^{-2}(E + \omega_0/2), \tag{8}$$

be an auxiliary spectral parameter. Then the Hamiltonian (5) has entire solutions with a finite Bargmann norm if and only if one of the following conditions is fulfilled. Either $x > 1$, and

$$\frac{(x-1)\left(\omega(\omega-\omega_0) + g^2(x-1)\right)}{\omega^2\sqrt{x^2-1}} = 2n+1, \tag{9}$$

or $x < -1$, and

$$\frac{(x-1)\left(\omega(\omega-\omega_0) + g^2(x-1)\right)}{\omega^2\sqrt{x^2-1}} = -(2n+1), \tag{10}$$

where $n \in \mathbb{N}$. The respective eigenfunctions

$$\psi_n^{\pm} = \left(\psi_{1,n}^{\pm}, \psi_{2,n}^{\pm}\right),$$

are given by

$$\begin{aligned} \psi_{1,n}^{\pm}(z) &= C\exp\left(-\beta_{\pm}z^2\right) H_n\left(\sqrt{\pm 1}\sqrt[4]{x^2-1}\, z\right), \\ \psi_{2,n}^{\pm}(z) &= \frac{\omega}{g(x-1)}\left[\psi_{1,n}^{\pm}(z) + \left(\psi_{1,n}^{\pm}(z)\right)'\right], \end{aligned} \tag{11}$$

where $H_n(z)$ denotes the Hermite polynomial of degree n, and

$$\beta_{\pm} := \frac{x \pm \sqrt{x^2-1}}{2}.$$

These functions have the growth order $\varrho = 2$ and type $\sigma = \beta_{\pm}$.