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Nonlinear vibration analysis of double-layered nanoplates with different boundary conditions

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ABSTRACT

By the nonlocal theory, the nonlinear equations for double-layered nanoplates (DLNP) with different boundary conditions are established. The relation between aspect ratio and nonlinear frequencies with fixed mode amplitude is discussed. This relation for two vibration modes presents completely distinct trends. The novel fact is observed that there exists a point *P* where nonlinearity is weakest for the fundamental mode. Furthermore, we notice that *P* only appears for clamped movable edge when neglecting the nonlocal effect, which is substantially different from the other boundary conditions. It should distinguish whether the edge is movable or immovable for studying nonlinear dynamics of DLNP.

1. Introduction

Since the popularity of carbon nanotubes and the pioneering work by Iijima [1], considerable researchers are engaged in the fundamental studies of nanostructures [2–5]. Nanoscience becomes a vigorous academic field [6–8]. With the superior performance for nanostructures over traditional engineering materials, many potential applications can be expected in atomic-force microscope, biosensors and field emitters [9,10]. Since the manufacturing and the design of nano devices for micro/nano electromechanical systems (MEMS/NEMS) heavily depend on the insights of the mechanical properties, researches have been performing widely on the dynamic behaviors of nanostructures [11–13].

With the difficulties in controlled experiments on nanoscale [14], theoretical analysis of mechanical behaviors of nanostructures is performed widely. However, molecular dynamics (MD) is time consuming and unable to deal with large-sized nanostructures such as nanoplates. Consequently, the nonlocal continuum theory initiated by Eringen [15,16] becomes an effective and reliable approach to model the nanostructures mathematically, which can accurately capture the small scale effect of nanoscale materials.

The mechanical behaviors of nanostructures have been the primary subjects of current studies. Reddy [17] presented the formulation of nonlinear dynamical equations for the classical and first shear deformation plates. Liew et al. [18] proposed a continuum-based plate model and derived the natural frequencies of multi-layered graphene sheets. Murmu and Pradhan [19] investigated the vibration response of single-layered graphene sheets embedded in elastic medium. The nonlinear dynamical properties of nanotubes have been studied by several researchers [20–23]. Wang and Li [24] investigated the static bending of nanoplates with the nonlocal Mindlin and Kirchhoff plate models. However, there are few reports about the nonlinear analysis of double-layered nanoplates (DLNP) comparing with abundant researches on the nonlinear dynamical behaviors of classical systems [25,26]. Recently, we investigated the nonlinear vibration properties of DLNP based on the nonlocal continuum theory [27].

In the present paper, based on the nonlocal theory, the nonlinear governing equations are derived for DLNP with four different boundary conditions. The analytical expressions of nonlinear vibration frequencies are obtained. We mainly discussed the relation between the aspect ratio and the nonlinear fundamental frequency for the DLNP with fixed mode amplitude. From the results, some novel phenomena can be observed. A minimum point *P* at which the nonlinearity is weakest can be found on the above-mentioned relation's curves. Furthermore, the existence and position of the point *P* are strongly dependent on the type of boundary conditions and the nonlocal parameter.

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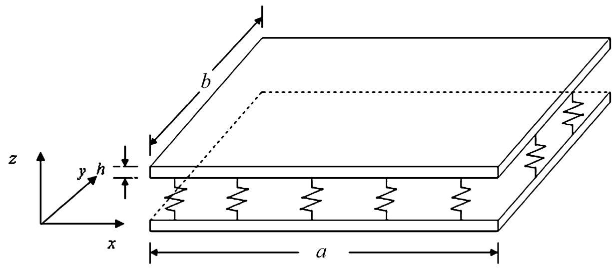

Fig. 1. Schematic diagram of double-layered nanoplates.

2. Dynamical equations of nonlinear vibration

The DLNP with the thickness h is shown in Fig. 1. According to the nonlocal continuum theory by Eringen [15,16], the constitutive relation of nonlocal elasticity is expressed as the following integral form:

$$\sigma_{kl}(x) = \int_V \alpha\left(\left|x - x'\right|, \gamma\right)\tau_{kl}\left(x'\right) dV\left(x'\right), \tag{1}$$

where σ_{kl} and τ_{kl} are the nonlocal stress tensors and local stress tensors, respectively. The kernel function $\alpha(|x - x'|, \gamma)$ is the nonlocal modulus which describes the influence of the strain at each point x' of the entire body on the stress of the reference point x. Considering the simplicity and the convenience for the applications on the elasticity problems, the kernel function is usually taken as

$$\alpha\left(|x|, \gamma\right) = \left(2\pi l^2\gamma^2\right)^{-1} K_0(\sqrt{x \cdot x}/l\gamma), \tag{2}$$

where K_0 is the modified Bessel function, l is the external characteristic length (e.g. crack length, wavelength), $\gamma = e_0 a_0/l$, where e_0 is a constant appropriate to each material and is determined by the experiments or atomic lattice dynamics, and a_0 the internal characteristic length (e.g. length of C–C bond, lattice spacing, granular distance).

For the nonlocal viscoelastic Kirchhoff's plate [27–30], the constitutive relation can be expressed as the following form:

$$\left(1 - \mu^2\nabla^2\right)\begin{Bmatrix}\sigma_{xx}^{nl}\\ \sigma_{yy}^{nl}\\ \sigma_{xy}^{nl}\end{Bmatrix} = \begin{bmatrix}\frac{E}{1-\upsilon^2}(1 + g\frac{\partial}{\partial t}) & \frac{\upsilon E}{1-\upsilon^2}(1 + g\frac{\partial}{\partial t}) & 0\\ \frac{\upsilon E}{1-\upsilon^2}(1 + g\frac{\partial}{\partial t}) & \frac{E}{1-\upsilon^2}(1 + g\frac{\partial}{\partial t}) & 0\\ 0 & 0 & 2G(1 + g\frac{\partial}{\partial t})\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{Bmatrix}, \tag{3}$$

where σ^{nl} is the nonlocal stress tensor, $\mu = e_0 a_0$ the nonlocal parameter which can describe the scale effect of nanostructures, E, G and υ denote the Young's modulus, the shear modulus and the Poisson's ratio, and g the viscoelastic structural damping coefficient.

The von Kármán nonlinear strain–displacement relation is employed herein [31,32]

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + z\boldsymbol{\kappa}, \tag{4}$$

where $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\kappa}$ are the strain vector and variation of curvature vector in middle surface and can be expressed as

$$\boldsymbol{\varepsilon}_0 = \left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]^T, \tag{5}$$

$$\boldsymbol{\kappa} = \left[-\frac{\partial^2 w}{\partial x^2}, -\frac{\partial^2 w}{\partial y^2}, -2\frac{\partial^2 w}{\partial x\partial y}\right]^T, \tag{6}$$

where u_0 and v_0 are the mid-plane displacements in the x and y directions, and w is the transverse displacement in the z direction.

For the DLNP system, the two plate layers bonded by van der Walls (vdW) force is usually modeled as the Winkler type foundation [33,34]. Then the distributed forces on the upper and lower plates, q_1' and q_2' are

$$q_1' = -c(w_1 - w_2), \tag{7a}$$

$$q_2' = -c(w_2 - w_1), \tag{7b}$$

where c is the vdW interaction coefficient, and w_1 and w_2 are the transverse displacements of the upper and lower plates.

Performing the Hamilton's principle and introducing the stress function F_1 and F_2 [35], the nonlinear equations of motion for the DLNP can be obtained as

$$\begin{aligned}&D\left(\frac{\partial^4 w_1}{\partial x^4} + 2\frac{\partial^4 w_1}{\partial x^2\partial y^2} + \frac{\partial^4 w_1}{\partial y^4} + g\frac{\partial^5 w_1}{\partial t\partial x^4} + 2g\frac{\partial^5 w_1}{\partial t\partial x^2\partial y^2} + g\frac{\partial^5 w_1}{\partial t\partial y^4}\right)\\ &\quad + (1 - \mu^2\nabla^2)\left[m_0\frac{\partial^2 w_1}{\partial t^2} - m_2\left(\frac{\partial^4 w_1}{\partial x^2\partial t^2} + \frac{\partial^4 w_1}{\partial y^2\partial t^2}\right)\right]\\ &\; = (1 - \mu^2\nabla^2)\left[-c(w_1 - w_2)\right] + (1 - \mu^2\nabla^2)\\ &\quad \times\left(\frac{\partial^2 F_1}{\partial y^2}\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial x^2}\frac{\partial^2 w_1}{\partial y^2} - 2\frac{\partial^2 F_1}{\partial x\partial y}\frac{\partial^2 w_1}{\partial y\partial x}\right),\\ &\nabla^4 F_1 = Eh\left[\left(\frac{\partial^2 w_1}{\partial y\partial x}\right)^2 - \left(\frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 w_1}{\partial y^2}\right)\right],\end{aligned} \tag{8a}$$

$$\begin{aligned}&D\left(\frac{\partial^4 w_2}{\partial x^4} + 2\frac{\partial^4 w_2}{\partial x^2\partial y^2} + \frac{\partial^4 w_2}{\partial y^4} + g\frac{\partial^5 w_2}{\partial t\partial x^4} + 2g\frac{\partial^5 w_2}{\partial t\partial x^2\partial y^2} + g\frac{\partial^5 w_2}{\partial t\partial y^4}\right)\\ &\quad + (1 - \mu^2\nabla^2)\left[m_0\frac{\partial^2 w_2}{\partial t^2} - m_2\left(\frac{\partial^4 w_2}{\partial x^2\partial t^2} + \frac{\partial^4 w_2}{\partial y^2\partial t^2}\right)\right]\\ &\; = (1 - \mu^2\nabla^2)\left[-c(w_2 - w_1)\right] + (1 - \mu^2\nabla^2)\\ &\quad \times\left(\frac{\partial^2 F_2}{\partial y^2}\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial x^2}\frac{\partial^2 w_2}{\partial y^2} - 2\frac{\partial^2 F_2}{\partial x\partial y}\frac{\partial^2 w_2}{\partial y\partial x}\right),\\ &\nabla^4 F_2 = Eh\left[\left(\frac{\partial^2 w_2}{\partial y\partial x}\right)^2 - \left(\frac{\partial^2 w_2}{\partial x^2}\frac{\partial^2 w_2}{\partial y^2}\right)\right],\end{aligned} \tag{8b}$$

where $D = Eh^3/12(1 - \upsilon^2)$ is the bending stiffness of the nanoplates, $m_0 = \int_{-h/2}^{h/2} \rho\, dz$ and $m_2 = \int_{-h/2}^{h/2} \rho z^2\, dz$, in which ρ denotes the density of the material. The equations can be readily degenerated to the classical cases by setting $\mu = 0$. The exhaustive derivation process can be referred to [27].

The boundary conditions for the remainder of the paper are related to the transverse displacement and the stress function. Specifically, four different boundary conditions are considered as follows.

I: Simply supported with movable edges

$$w = \frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 F}{\partial x\partial y} = \int_0^b \frac{\partial^2 F}{\partial y^2}\, dy = 0, \quad \text{at } x = 0, a, \tag{9a}$$

$$w = \frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 F}{\partial x\partial y} = \int_0^a \frac{\partial^2 F}{\partial x^2}\, dx = 0, \quad \text{at } y = 0, b. \tag{9b}$$