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The non-singular Green tensor of Mindlin's anisotropic gradient elasticity with separable weak non-locality

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ABSTRACT

In this paper, we derive the Green tensor of anisotropic gradient elasticity with separable weak non-locality, a special version of Mindlin's form II anisotropic gradient elasticity theory with up to six independent length scale parameters. The framework models materials where anisotropy is twofold, namely the bulk material anisotropy and a weak non-local anisotropy relevant at the nano-scale. In contrast with classical anisotropic elasticity, it is found that both the Green tensor and its gradient are non-singular at the origin, and that they rapidly converge to their classical counterparts away from the origin. Therefore, the Green tensor of Mindlin's anisotropic gradient elasticity with separable weak non-locality can be used as a physically-based regularization of the classical Green tensor for materials with strong anisotropy.

1. Introduction

Mindlin's theory of strain gradient elasticity [1–3] is a well-suited framework to model the behavior of elastic materials at the nano-scale. In fact, using ab initio calculations, Shodja et al. [4] found that the characteristic length scale parameters of Mindlin's gradient elasticity theory are in the order of $\sim 10^{-10}$ m for several fcc and bcc materials. Therefore, as a generalization of classical elasticity, gradient elasticity becomes relevant for nano-mechanical phenomena at such length scales. However, the most general version of Mindlin's strain gradient elasticity has found limited application because of both its complexity and the presence of a large number of new material parameters.

A simplified version of Mindlin's gradient elasticity with only one gradient parameter, known as gradient elasticity of Helmholtz type, has successfully been used to model non-singular straight dislocations [5] and dislocation loops [6–8] in isotropic materials. Po et al. [9] have further developed the aspects of the theory relevant to its numerical application to Dislocation Dynamics simulations. In a recent paper [10], the isotropic theory of gradient elasticity of Helmholtz type was generalized towards gradient anisotropic elasticity of Helmholtz type. In particular, Lazar and Po [10] have derived the Green tensor of gradient anisotropic elasticity of Helmholtz type. The Green tensor and its gradient were found to be non-singular at the origin, in contrast with their classical counterparts (e.g., [11–18]). In addition, it was shown that the Green tensor and its gradient rapidly converge to their classical limits a few characteristic lengths away from the origin. Therefore, the Green tensor of gradient anisotropic elasticity of Helmholtz type can be used as a physically-based regularization of the classical anisotropic Green tensor for applications in problems of nano-mechanics of materials and their defects (e.g., [19]). Having only one characteristic length scale parameter, gradient anisotropic elasticity of Helmholtz type models materials which are anisotropic in the elastic material moduli, but possess an isotropic weak non-locality. As discussed later in this paper, cubic materials belong to this category.

For materials with a more general degree of anisotropy, however, there is no particular reason to assume that the weak non-locality possesses an isotropic character. Therefore, modeling of such materials in the framework of Mindlin's form II gradient elasticity requires a generalization of gradient anisotropic elasticity of Helmholtz type. In this paper, we introduce a framework which we call Mindlin's anisotropic gradient elasticity with separable weak non-locality. Such a theory is a special version of Mindlin's anisotropic gradient elasticity where the sixth-rank tensor of strain gradient coefficients is decomposed into the product of the fourth-rank tensor of elastic coefficients and a second-rank tensor of characteristic length scale parameters. Thus, this framework is a generalization of gradient anisotropic Helmholtz elasticity because

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its governing equation involves an anisotropic Helmholtz operator with up to six independent length scale parameters. The presence of multiple length scales gives rise to a weak non-locality with anisotropic character. Therefore, in Mindlin's anisotropic gradient elasticity with separable weak non-locality, anisotropy has two distinct sources, namely the bulk anisotropy and the weak non-local anisotropy.

The fundamental result of this paper consists in the determination of the Green tensor (fundamental solution) of Mindlin's anisotropic gradient elasticity with separable weak non-locality. The Green tensor is non-singular at the origin, and it rapidly converges to the Green tensor of classical anisotropic elasticity away from the origin. Therefore, the non-singular Green tensor can be used as physically-based regularization of the classical Green tensor in problems of nano-mechanics of materials with strong anisotropy.

This paper is organized as follows. In Section 2, we derive the theory of Mindlin's anisotropic gradient elasticity with separable weak non-locality from the general Mindlin's form II anisotropic gradient elasticity theory. In Section 3, we construct the Green tensor of the twofold anisotropic Helmholtz–Navier operator appearing in the partial differential equation representing the equilibrium condition of the theory. Using the Fourier transform method, the non-singular anisotropic Green tensor is obtained as a surface integral on the unit sphere. In addition, we derive expressions for the gradient of the Green tensor. Discussion and conclusions are presented in section conclusions. In Appendix A, we give the form of the second-rank tensor of characteristic length scale parameters in relationship to different classes of material symmetries. In Appendix B, we derive the Green function of the anisotropic Helmholtz operator which constitutes part of the theory and may be used for its future applications in the nano-mechanics of materials and their defects.

2. Mindlin's anisotropic gradient elasticity with separable weak non-locality

Consider an infinite elastic body in three-dimensional space and assume that the gradient of the displacement field $\boldsymbol{u}$ is additively decomposed into an elastic distortion tensor $\boldsymbol{\beta}$ and an inelastic[1] distortion tensor $\boldsymbol{\beta}^*$:

$$\partial_j u_i = \beta_{ij} + \beta^*_{ij} \,. \tag{1}$$

The elastic strain tensor, e_{ij}, is the symmetric part of β_{ij}:

$$e_{ij} = \frac{1}{2}\left(\beta_{ij} + \beta_{ji}\right) . \tag{2}$$

In Mindlin's form II of anisotropic gradient elasticity theory [1–3], the strain energy density for a homogeneous and centrosymmetric[2] material is given by (see also [20])

$$\mathcal{W} = \frac{1}{2}\, C_{ijkl} e_{ij} e_{kl} + \frac{1}{2}\, D_{ijmkln} \partial_m e_{ij} \partial_n e_{kl} \,, \tag{3}$$

where C_{ijkl} is the standard fourth-rank tensor of elastic constants for an anisotropic material with symmetry properties

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \,, \tag{4}$$

possessing 21 independent components for general anisotropy (triclinic), while D_{ijmkln} is a sixth-rank constitutive tensor which accounts for anisotropic weak non-locality and possesses the following symmetries

$$D_{ijmkln} = D_{jimkln} = D_{ijmlkn} = D_{klnijm} \,. \tag{5}$$

In the fully anisotropic (triclinic) version of Mindlin's strain gradient elasticity theory, the number of independent components of the tensor D_{ijmkln} is equal to 171 (see [21]). We shall now assume (see also [10,20,22]) that the sixth-rank tensor D_{ijmkln} can be decomposed into the product of the fourth-rank tensor C_{ijkl} and a second-rank tensor Λ_{mn} of gradient length scale parameters with units of squared length, that is

$$D_{ijmkln} = C_{ijkl} \Lambda_{mn} \,. \tag{6}$$

As a consequence of the symmetry properties (5) and of the positive definiteness of $\mathcal{W}$, the tensor Λ_{mn} must be symmetric and positive definite:

$$\Lambda_{mn} = \Lambda_{nm} \tag{7}$$

$$x_m \Lambda_{mn} x_n > 0 \qquad \forall \boldsymbol{x},\ \|\boldsymbol{x}\| > 0 \,. \tag{8}$$

By virtue of the decomposition (6), the 192 $(21 + 171)$ independent material constants of a centrosymmetric triclinic material in Mindlin's anisotropic gradient elasticity theory are reduced to 21 elastic constants and 6 length scale parameters in Mindlin's anisotropic gradient elasticity with separable weak non-locality. For materials with a lower degree of anisotropy, both the tensor C_{ijkl} and the tensor Λ_{mn} must fulfill restrictions resulting from point symmetries of the specific material. These restrictions are considered in Appendix A for triclinic, monoclinic, orthorhombic, tetragonal, hexagonal, trigonal, cubic, and isotropic materials.

From a physical viewpoint, the decomposition (6) represents the separation of the two sources of anisotropy present in Mindlin's anisotropic gradient elasticity, namely the elastic bulk anisotropy and the anisotropy of the gradient length scale parameters (weak non-local anisotropy). The latter, which is not present in classical anisotropic elasticity, reflects the discrete nature of matter and becomes relevant in the presence of structures and defects of comparable sizes. Therefore, the anisotropy of the length scale parameters gives rise to new physical anisotropic effects[3] at the nano-scale.

Under the above assumptions, the strain energy density (3) of the anisotropic body reads

$$\mathcal{W} = \frac{1}{2}\, C_{ijkl} e_{ij} e_{kl} + \frac{1}{2}\, \Lambda_{mn} C_{ijkl} e_{ij,m} e_{kl,n} \,. \tag{9}$$

The quantities conjugate to the elastic strain tensor and its gradient are the Cauchy stress tensor $\boldsymbol{\sigma}$ and the double stress tensor $\boldsymbol{\tau}$, respectively. These are defined as:

$$\sigma_{ij} = \frac{\partial \mathcal{W}}{\partial e_{ij}} = C_{ijkl} e_{kl} \,, \tag{10}$$

$$\tau_{ijk} = \frac{\partial \mathcal{W}}{\partial(\partial_k e_{ij})} = \Lambda_{kl}\, C_{ijmn} e_{mn,l} = \Lambda_{kl} \sigma_{ij,l} \,. \tag{11}$$

In the presence of a body forces density $\boldsymbol{b}$, the static Lagrangian density of the system becomes:

$$\begin{aligned} \mathcal{L} &= -\mathcal{W} - \mathcal{V} \\ &= -\left(\frac{1}{2}\, C_{ijkl} \beta_{ij} \beta_{kl} + \frac{1}{2}\, \Lambda_{mn} C_{ijkl} \beta_{ij,m} \beta_{kl,n}\right) + u_i b_i \,, \end{aligned} \tag{12}$$

[1] The inelastic distortion comprises plastic and thermal effects, and is typically an incompatible field. When the inelastic distortion is absent the elastic distortion is compatible.

[2] Due to the centrosymmetry, there is no coupling between e_{ij} and $\partial_m e_{kl}$.

[3] The concept of non-local anisotropy is also present in Eringen's theory of nonlocal anisotropic elasticity [23] (see also [24]). Lazar and Agiasofitou [24] show how anisotropic non-locality can be modeled by means of an anisotropic Helmholtz operator.