



Fractional optimal control problem of a distributed system in cylindrical coordinates

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ABSTRACT

In this work, Fractional Optimal Control Problem (FOCP) of a Distributed system is investigated in cylindrical coordinates. Axis-symmetry naturally arises in the problem formulation. The fractional time derivative is described in the Riemann–Liouville (RL) sense. The performance index of a FOCP is considered as a function of state and control variables and system dynamics are given as a Partial Fractional Differential Equation (PFDE). The method of separation of variables is used to find the solution of the problem. Eigenfunctions are used to eliminate the terms containing space parameters and to define the problem in terms of a set of generalized state and control variables. For numerical computations, Grünwald–Letnikov (GL) approach is used. A time-invariant example is considered to demonstrate the effectiveness of the formulation. The comparison of analytical and numerical solutions is given using simulation results and also it can be seen that analytical and numerical results converge each other. In addition, simulation results for different values of order of derivative, time discretizations and eigenfunctions are analyzed.

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1. Introduction

In the last years, it has been showed that the accurate modelling of dynamics of many physical systems can be obtained using Fractional Differential Equations (FDEs). Therefore, there has been a great deal of interest in the solution methods of FDEs in analytical and numerical sense. When FDEs describe the performance index and system dynamics, an optimal control problem reduces to a FOCP. The Fractional Optimal Control (FOC) of a distributed system is a FOC for which system dynamics are defined with PFDEs. There has been very little work in the area of FOCPs, especially FOC of a distributed system.

In the area of Fractional Order Controls and Systems, there are some papers which must be mentioned here. Oustaloup [1] investigated fractional order controls for dynamic systems and showed that the CRONE method has better performance than classical PID controller. Podlubny [2] demonstrated that PI , PD and PID controllers are particular cases of the fractional $PI^\lambda D^\mu$ controller. Podlubny, Dorcak and Kostial [3] compared RL–GL and Caputo fractional derivatives from the viewpoint of formulation and solution of engineering and physics problems, and they also presented the fractional $PI^\lambda D^\mu$ controller. Dorcak [4] analyzed dynamic prop-

erties and numerical methods of simulation of fractional-order systems. Petras, Dorcak and Kostial [5] dealt with fractional-order controlled systems and fractional-order controllers in discrete time domain. Machado [6,7] introduced algorithms for fractional-order discrete time controllers. Özdemir and İskender [8] applied fractional PI^λ controller for fractional order linear system subject to input hysteresis. Although, these cited papers show that the research area of fractional-order systems and controllers is popular, they do not mention FOCPs.

Recently, some papers related to the theories and solution methods of FOCPs have been presented. A general formulation and a numerical scheme for FOCPs in RL sense are investigated in Agrawal [9]. Agrawal [10] presents an eigenfunction expansion approach for a FOCP for a class of distributed system whose dynamics are defined in Caputo sense. Özdemir et al. [11] also use eigenfunction expansion approach to formulate a FOCP of a 2-dimensional distributed system. A general scheme for stochastic analysis of FOCPs is proposed in Agrawal [12]. A formulation for FOCPs whose dynamics are described in terms of Caputo fractional derivative is researched in Agrawal [13,14] and the same problem is investigated in terms of RL fractional derivatives in Agrawal and Baleanu [15].

In this Letter, we formulate a FOCP of a 3-dimensional distributed system defined in cylindrical coordinates. For this reason, the axis-symmetric case arises naturally in this problem. It is formulated in terms of RL fractional derivative and GL approach is used for numerical computation. Özdemir et al. [11] consider the problem in 2-dimensional case (Cartesian coordinates), whereas

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in this Letter we formulate the problem in 3-dimensional case (cylindrical coordinates). The solution of problem is obtained for different number of eigenfunctions and time discretization. Also, the papers related to axis-symmetry can be given as follows: El-Shahed and Salem [16] find the solution of fractional generalization of Navier–Stokes equations described by polar coordinates. Fractional radial diffusion in a cylinder and in a sphere are proposed in Povstenko [17,18], respectively. Özdemir et al. [19], and Özdemir and Karadeniz [20] have recently formulated an axis-symmetric fractional diffusion-wave problem.

This Letter is organized as follows. In Section 2, the definitions of RL fractional derivative and FOCF are given. In Section 3, an axis-symmetric FOCF defined in cylindrical coordinates is presented. In Section 4, the GL approach is given and numerical results are analyzed. In Appendix A, the analytical solution of the problem is presented. Finally, Section 5 shows conclusions of this work.

2. Mathematical tools

Several definitions of a fractional derivative such as Riemann–Liouville, Caputo, Grünwald–Letnikov, Weyl, Marchaud and Riesz have been proposed. In this section, we formulate the problem in terms of the Riemann–Liouville fractional derivatives, which are defined as:

The left Riemann–Liouville fractional derivative

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad (1)$$

and the right Riemann–Liouville fractional derivative

$${}_t D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt} \right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau, \quad (2)$$

where $f(\cdot)$ is a time-dependent function, $\Gamma(\cdot)$ is the Euler's gamma function, α is the order of derivative such that $n-1 < \alpha < n$ and t represents time which belongs to $[a, b]$ ($a, b \in \mathbb{R}$). When α is an integer, these definitions reduce to ordinary differential operators, i.e.,

$$\begin{aligned} {}_a D_t^\alpha f(t) &= \left(\frac{d}{dt} \right)^n, \\ {}_t D_b^\alpha f(t) &= \left(-\frac{d}{dt} \right)^n, \quad \alpha = n, \quad n = 1, 2, \dots \end{aligned} \quad (3)$$

Using the above definitions, the FOCF of interest is defined in Agrawal [9,15] as follows: Find the optimal control $u(t)$ that minimizes the performance index

$$J(u) = \int_0^1 F(x, u, t) dt \quad (4)$$

subject to the system dynamic constraints

$${}_0 D_t^\alpha x = G(x, u, t) \quad (5)$$

and initial condition

$$x(0) = x_0, \quad (6)$$

where $x(t)$ and $u(t)$ are the state and the control variables, respectively, F and G are two arbitrary functions. For $\alpha = 1$, this problem reduces to a standard optimal control problem. Moreover, we take $0 < \alpha < 1$ and assume that $x(t)$, $u(t)$ and $G(x, u, t)$ are all scalar functions for simplicity. In the case of $\alpha > 1$, additional initial conditions could be necessary. In the usual sense, the differential equations which describe the dynamics of the system are written

in the state-space form, in that case, the order of the derivatives turns out to be less than 1. For this reason, we consider $0 < \alpha < 1$ in this work. We further consider the necessary terminal conditions which is determined by using Lagrange multiplier technique as follows:

$${}_0 D_t^\alpha x = G(x, u, t), \quad (7)$$

$${}_t D_1^\alpha \lambda = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial x} \lambda, \quad (8)$$

$$\frac{\partial F}{\partial u} + \frac{\partial G}{\partial u} \lambda = 0, \quad (9)$$

where λ is the Lagrange multiplier and

$$x(0) = x_0 \quad \text{and} \quad \lambda(1) = 0. \quad (10)$$

Eqs. (7)–(9) represent the Euler–Lagrange equations for the FOCF defined by Eqs. (4)–(6). This indicates that the solution of FOCFs requires not only right derivatives but also left derivatives.

3. The axis-symmetric FOCF formulation

Let us consider the following problem: Find the control $u(r, z, t)$ that minimizes the performance index

$$J(u) = \frac{1}{2} \int_0^1 \int_0^L \int_0^R r [Ax^2(r, z, t) + Bu^2(r, z, t)] dr dz dt \quad (11)$$

subject to the system dynamic constraints

$$\begin{aligned} {}_0 D_t^\alpha x(r, z, t) &= \beta \left(\frac{\partial^2 x(r, z, t)}{\partial r^2} + \frac{1}{r} \frac{\partial x(r, z, t)}{\partial r} + \frac{\partial^2 x(r, z, t)}{\partial z^2} \right) \\ &\quad + u(r, z, t), \end{aligned} \quad (12)$$

initial condition

$$x(r, z, 0) = x_0(r, z) \quad (0 < r < R, \quad 0 < z < L) \quad (13)$$

and the boundary conditions

$$\frac{\partial x(0, z, t)}{\partial r} = \frac{\partial x(R, z, t)}{\partial r} = \frac{\partial x(r, 0, t)}{\partial z} = \frac{\partial x(r, L, t)}{\partial z} = 0, \quad (14)$$

where $x(r, z, t)$ and $u(r, z, t)$ are the state and the control functions depending on r, z , which represent cylindrical coordinates, and t . A and B are two arbitrary functions. R is the radius and L is the length of cylindrical domain on which problem is defined. The upper limit for time t is taken as 1 for convenience. This limit can be any positive number.

We assume that $x(r, z, t)$ and $u(r, z, t)$ can be written as

$$x(r, z, t) = \sum_{i=1}^m \sum_{j=1}^m x_{ij}(t) J_0 \left(\mu_j \frac{r}{R} \right) \sin \left(\frac{i\pi}{L} z \right), \quad (15)$$

$$u(r, z, t) = \sum_{i=1}^m \sum_{j=1}^m u_{ij}(t) J_0 \left(\mu_j \frac{r}{R} \right) \sin \left(\frac{i\pi}{L} z \right), \quad (16)$$

where $J_0(\mu_j \frac{r}{R}) \sin(\frac{i\pi}{L} z)$, $i, j = 1, 2, \dots, m$, are the eigenfunctions which are obtained by using the method of separation of variables. Here, J_0 is zero order Bessel function of first kind and μ_j are the roots of J_0 . $x_{ij}(t)$ and $u_{ij}(t)$ are the state and the control eigencoordinates; m is a finite positive integer that theoretically should go to infinity. However, we take m as a finite number for computational purposes. By substituting Eqs. (15) and (16) into Eq. (11), we obtain

$$J = \frac{R^2 L}{8} \int_0^1 \sum_{i=1}^m \sum_{j=1}^m J_1^2(\mu_j) [Ax_{ij}^2(t) + Bu_{ij}^2(t)] dt. \quad (17)$$

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