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Consensus states of local majority rule in stochastic process

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ABSTRACT

A sufficient condition for a network system to reach a consensus state of the local majority rule is shown. The influence of interpersonal environment on the occurrence probability of consensus states for Watts–Strogatz and scale-free networks with random initial states is analyzed by numerical method. We also propose a stochastic local majority rule to study the mean first passage time from a random state to a consensus and the escape rate from a consensus state for systems in a noisy environment. Our numerical results show that there exists a window of fluctuation strengths for which the mean first passage time from a random to a consensus state reduces greatly, and the escape rate of consensus states obeys the Arrhenius equation in the window.

1. Introduction

Consider a group of individuals in a process of forming opinions on an issue. Individual opinions generally have a wide spectrum and they are liable to change owing to mutual influences among individuals. The process of opinion dynamics may eventually lead to a consensus among the individuals, or to a certain fragmental pattern of opinions, or to a periodic oscillation between configurations of opinions. Many models have been constructed and analyzed to answer the question of how individual opinions may form in a complex interpersonal environment, in particular, that of how a group consensus is reached has attracted great research interest [1–14]. In theoretical study, the interpersonal environment can be visualized as a network in which, the nodes represent individuals and the edges between the nodes signify the mutual social influences between the connected pairs. Opinion space can be either discrete or continuous. Physics community, motivated by the research in spin systems, often take discrete opinion spaces for the study [1–3]. The local majority rule (*LMR*) uses binary opinions to assign two states, $+1$ and -1, to a node, and its dynamics specifies a node-state in the next time-step as the state possessed by the majority of its neighbors [15,16]. One of the equilibrium states for the dynamics is a group consensus, which is a configuration that all node-states are the same. In this paper, we analyze the relation between the occurrence of a group consensus and the interpersonal environment for the opinion dynamics of *LMR*.

Although *LMR* is simple for only considering the environmental or social preference from the neighbors, the dynamics is highly nonlinear. In the analysis we first show a sufficient condition for reaching a consensus; however, a necessary and sufficient condition cannot be obtained. Thus, to gain more understandings about the effect of interpersonal environment for reaching a consensus, we employ statistical measures to study the occurrence of a consensus state including stochastic process. The change of a node state depends only on the states possessed by the connected neighbors for *LMR*; one can expect that the geometric distribution of edges plays an important role in the type of equilibrium states reached by a system [17,18]. As the geometric structure of a network can be described by its connection matrix, we introduce proper parameters to characterize a set of connection matrices of Watts–Strogatz (*WS*) or scale-free (*SF*) networks [19–23]. Then, the members belonging to a set have the same global geometric property, we calculate numerically the occurrence probability of reaching a consensus for a set of connection matrices with random initial states, and the results are analyzed to obtain the geometric effect on the reachability of consensus states for *WS* and *SF* networks.

Another issue related to the occurrence of a consensus state is that how an interpersonal environment with fluctuation affects the reachability and the stability of a consensus state. Moreira et al. used a mean-field approach to show that the presence of fluc-

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tuation may increase the occurrence probability and decrease the first-passage time of a consensus state for systems with small-world characters [24]. Other form of stochastic *LMR* was also proposed in the literatures [25–27]. To have a detailed understanding on this issue, we propose a stochastic *LMR* in which, a component of white noise is added to a node-state. Then, the stochastic *LMR* is employed to study the mean first-passage time from a random state to a consensus state, and a window of fluctuation strength, which makes the mean first passage-time to be much shorter than the result of no fluctuation, is found. Moreover, the escape rate of a consensus state is shown to obey the Arrhenius equation when fluctuation strength is in the window, and the geometric influence on the prefactor and the activation energy of the Arrhenius equation is also discussed.

This paper is organized as follows. In Section 2, we define *LMR*, give different types of equilibrium states, and show that an irreducible and primitive connection matrix of a network is a sufficient condition for reaching a consensus. In Section 3, we first construct different sets of connection matrices of *WS* and *SF* networks, each set is specified by a global geometric property. Then, the occurrence probability of consensus states in a set of connection matrices with random initial states is calculated, and the geometric effect on the occurrence probability is analyzed. In Section 4, a stochastic *LMR* is proposed to study the property of consensus states when fluctuation appears. The numerical results for the mean first-passage time from a random configuration to a consensus state and the escape rate of a consensus state are given and analyzed. Finally, we summarize the results in Section 5.

2. Local majority rule and equilibrium states

We first specify the *LMR* and classify its equilibrium states. Consider a network system with the distribution of edges between nodes given by an $N \times N$ connection matrix Γ. Here the entries of the matrix Γ are given as $\gamma_{ij} = 1$ for the connected nodes i and j, and 0 otherwise. The dynamic variable associated with node i at time t is denoted as $x_i(t)$, which takes discrete values, either 1 or -1. The system evolves from an initial to a new configuration in discrete time step according to *LMR* whose operation can be either synchronous or asynchronous. In this work, we consider the synchronous dynamics for which, the rule can be written as

$$x_i(t+1) = \mathbf{sgn}\left(\sum_{j=1}^{N} \gamma_{ij} x_j(t)\right) \tag{1}$$

for $i = 1, \ldots, N$, where the **sgn** function is a standard threshold function with $\mathbf{sgn}(x) = +1$ for $x > 0$ and -1 for $x < 0$, and we set $x_i(t+1) = x_i(t)$ for $\sum_{j=1}^{N} \gamma_{ij} x_j(t) = 0$. The rule has been applied widely to the study of discrete neural networks. One can use the Lyapunov energy function,

$$E(t) = -\sum_{i=1}^{N} x_i(t) \left[\sum_{j=1}^{N} \gamma_{ij} x_j(t-1)\right], \tag{2}$$

to show that there exists equilibrium states with the period 1 or 2 [15,16]. For the equilibrium states of period-1, there are two types of configurations: One is consensus states for which, all nodes are in a same state, 1 or -1; the other is partial consensus states for which, only part of the nodes are in the same state. For those with period-2, the system oscillates between a pair of configurations.

The structure of connection matrix Γ is crucial for the type of equilibrium states reached by a trajectory, we then analyze the relation between Γ and the equilibrium state reached by a system. By introducing $X(k) = (x_1(k), x_2(k), \ldots, x_N(k))^\tau$ with the superscript τ for the transpose, we rewrite Eq. (1) as

$$X(k+1) = \phi\big(\Gamma \cdot X(k)\big) \tag{3}$$

for which, the components are given as

$$x_i(k+1) = \phi\left(\sum_{j=1}^{N} \gamma_{ij} x_j(k)\right). \tag{4}$$

Here, $\phi(x)$ is the same as the function $\mathbf{sgn}(x)$ of Eq. (1) for $x \neq 0$, but it has the value $\phi(x) = 1$ at the point $x = 0$ for the convenience of analysis without losing any generality.

A nonnegative Γ is either reducible or irreducible. For a reducible Γ, we can relabel the nodes to represent Γ as the union of disjoint submatrices [28]. For example, Γ may consist of two submatrices,

$$\Gamma = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}, \tag{5}$$

where Γ_1 is $n_1 \times n_1$ and Γ_2 is $n_2 \times n_2$ matrices with $n_1 + n_2 = N$. Because of lacking communications between submatrices, a consensus state cannot be guaranteed to be reached by a system.

Suppose that Γ is irreducible. Then, we can classify Γ into two types, primitive and imprimitive, based on the Perron–Frabenius theorem [28]. As Γ is primitive, there exists an integer n that $(\Gamma)^n$ is positive, that is, $((\Gamma)^n)_{ij} > 0$ for $i, j = 1, \ldots, N$. We observe that

$$\phi\big(\Gamma \cdot \phi(\Gamma \cdot X)\big) \geq \phi\big(\Gamma^{\otimes 2} \cdot X\big) \tag{6}$$

with $\Gamma^{\otimes 2} = \Gamma \otimes \Gamma$, where the symbol $\otimes$ represents the Boolean product which is defined as $(\Gamma \otimes \Gamma)_{ij} = 1$ for $(\Gamma \cdot \Gamma)_{ij} \geq 1$ and 0 otherwise. Then, by applying the inequality of Eq. (6) n times successively, we have

$$X(n+1) = \phi\Big(\Gamma\Big(\cdot\cdot\phi\Big(\Gamma\phi\big(\Gamma X(0)\big)\Big)\Big)\cdot\cdot\Big) \geq \phi\big(\Gamma^{\otimes n} X(0)\big). \tag{7}$$

Since $(\Gamma)^n$ is positive, we have $\Gamma^{\otimes n} = [1]$, where [1] is the $N \times N$ matrix with 1 for all entries. Consequently, we have

$$X(n+1) = \pm I_N, \tag{8}$$

where I_N is the N-dimensional column vector with 1 for all entries, and the plus (minus) sign on the right hand side is taken for $\sum_{i=1}^{N} x_i(0) \geq 0$ ($\sum_{i=1}^{N} x_i(0) < 0$). Thus, every trajectory will be leaded to a consensus state for a primitive Γ. However, the condition of primitive Γ is sufficient but not necessary, as an explicit example shown below that a trajectory may also be leaded to a consensus state for an imprimitive Γ.

Consider the case of imprimitive Γ. As the edges between nodes are undirected in a network, the matrix Γ is symmetric, and we have $-\rho(\Gamma) \in \sigma(\Gamma)$ and $|-\rho(\Gamma)| = \rho(\Gamma)$, where $\sigma(\Gamma)$ and $\rho(\Gamma)$ are the spectrum and the spectrum radius of Γ, respectively. This gives the index of imprimitivity to be 2. Then, we can relabel the nodes to have Γ in the form of

$$\Gamma = \begin{pmatrix} 0 & \Gamma_1 \\ \Gamma_2 & 0 \end{pmatrix}, \tag{9}$$

where Γ_1 and Γ_2 are $n_1 \times n_2$ and $n_2 \times n_1$ matrix, respectively, with $n_1 + n_2 = N$. Based on this form, we consider an explicit Γ for which, the submatrices Γ_1 and Γ_2 are

$$\Gamma_1 = (\Gamma_2)^\tau = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{10}$$

This may lead systems to a variety of equilibrium states for different initial states, including two consensus states, two partial consensus states, and 6 distinct cycles of period 2. In fact, among the total 256 configurations for the initial states of trajectories, there are 89 for reaching a consensus state, 8 for a partial consensus, and 159 for a cycle of period 2.