Physics Letters A 372 (2008) 4768–4774

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Permutation entropy of fractional Brownian motion and fractional Gaussian noise

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ARTICLE INFO

Article history:
Received 24 September 2007
Received in revised form 14 April 2008
Accepted 6 May 2008
Available online 17 May 2008
Communicated by A.R. Bishop

PACS:
02.50.-r
02.50.Ey
05.45.Tp
89.70.Cf

Keywords:
Permutation entropy
Fractional Brownian motion
Fractional Gaussian noise

ABSTRACT

We have worked out theoretical curves for the permutation entropy of the fractional Brownian motion and fractional Gaussian noise by using the Bandt and Shiha [C. Bandt, F. Shiha, J. Time Ser. Anal. 28 (2007) 646] theoretical predictions for their corresponding relative frequencies. Comparisons with numerical simulations show an excellent agreement. Furthermore, the entropy-gap in the transition between these processes, observed previously via numerical results, has been here theoretically validated. Also, we have analyzed the behaviour of the permutation entropy of the fractional Gaussian noise for different time delays.

1. Introduction

Entropic studies almost always assume that the underlying probability distribution is given. This is not at all the case if one is dealing with an input signal regarded as a time series. Part of the concomitant analysis involves extracting the probability distribution from the data and, rarely, a univocal procedure imposes itself. One recent and successful method is that introduced by Bandt and Pompe [1]. The Bandt and Pompe method (BPM) for evaluating the probability distribution is based on the details of the attractor reconstruction procedure. It is the only one among those in popular use that takes into account the temporal structure of the time series generated by the physical process under study. A notable result from the Bandt and Pompe approach is a notorious improvement in the performance of the information quantifiers obtained using the probability distribution generated by their algorithm [2–8]. Of course, one must assume with the BPM that the system fulfills a very weak stationary condition and that enough data are available for a correct attractor reconstruction. The *permutation entropy* is just the celebrated Shannon entropic measure evaluated using the BPM to extract the associated probability distribution.

We are interested in the characterization of stochastic processes through this quantifier. In particular, we have chosen the fractional Brownian motion and its noise, the fractional Gaussian noise, for the analysis. The former is a ubiquitous non-stationary model for many physical phenomena which have empirical spectra of power-law type, $1/f^{\alpha}$, with $1 < \alpha < 3$. Thus, the characterization of these processes has become of interest in different and heterogeneous scientific fields, like physics, biology, finance, telecommunications and music [9–12]. It should be stressed that both processes, fBm and fGn, were jointly introduced in the seminal work of Mandelbrot and Van Ness published in 1968 [13]. Moreover, many authors have made use of the physical connection between fBm and fGn for modelling and synthesis purposes [14–17].

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.05.026

In a previous effort [18], the normalized permutation entropy of the fractional Gaussian noise and fractional Brownian motion was numerically computed. A clear entropy-gap was observed in the transition between these two stochastic processes, that does not depend upon neither the length of the associated time series nor the embedding dimension. Curiously enough, this is a *new* result. Previous approaches that employ probability distributions based on a wavelet description fail to detect such gap [19].

In this Letter we have worked out theoretical curves for the above-mentioned normalized permutation entropy of the fractional Gaussian noise and fractional Brownian motion. To such an end we have used theoretical results published recently by Bandt and Shiha [20]. This allows the previously observed entropy-gap to be now conclusively classified as a real phenomenon, not a numerical artifact. Also, we have analyzed the behaviour of the normalized permutation entropy of the fractional Gaussian noise for different time delays. Finally, the curves we worked out using the Bandt and Shiha results were compared with those obtained from numerical simulations of the two stochastic processes under analysis.

The reminder of the Letter is organized as follows. In Section 2 we describe the Bandt and Pompe probability distribution and its associated permutation entropy. In Section 3 we give a brief review of the two stochastic processes under analysis: the fractional Gaussian noise and fractional Brownian motion. The theoretical curves and the comparison with their numerical simulations counterparts are presented in Section 4. Discussions and conclusions are the subject of the last section. Finally, in Appendix A we give some details concerning the Bandt and Shiha theoretical results that we use throughout the Letter.

2. The Bandt and Pompe approach

Given a time series $\{x_t\colon\ t = 1, \ldots, M\}$, an embedding dimension $D > 1$, and a time delay τ, consider the ordinal patterns of order D [1,2,21] generated by

$$s \mapsto (x_{s-(D-1)\tau}, x_{s-(D-2)\tau}, \ldots, x_{s-\tau}, x_s). \tag{1}$$

To each time s we are assigning a D-dimensional vector that results from the evaluation of the time series at times $s, s-\tau, \ldots, s-(D-1)\tau$. Clearly, the greater the D value, the more information about the past is incorporated into the ensuing vectors. By the ordinal pattern of order D related to the time s we mean the permutation $\pi = (r_0, r_1, \ldots, r_{D-1})$ of $(0, 1, \ldots, D-1)$ defined by

$$x_{s-r_{D-1}\tau} \leqslant x_{s-r_{D-2}\tau} \leqslant \cdots \leqslant x_{s-r_1\tau} \leqslant x_{s-r_0\tau}. \tag{2}$$

In order to get a unique result we consider that $r_i < r_{i-1}$ if $x_{s-r_i\tau} = x_{s-r_{i-1}\tau}$. Thus, for all the $D!$ possible permutations π_i of order D, the probability distribution $P = \{p(\pi_i), i = 1, \ldots, D!\}$ is given by the relative frequency

$$p(\pi_i) = \frac{\sharp\{s \mid s \geqslant 1 + (D-1)\tau, s \text{ has ordinal pattern } \pi_i\}}{M - (D-1)\tau}, \tag{3}$$

where $\sharp$ is the cardinality of the set—roughly speaking, the number of elements in it. To determine $p(\pi_i)$ exactly an infinite time series should be considered, taking $M \to \infty$ in the above formula. This limit exists with probability 1 when the underlying stochastic process fulfills a very weak stationarity condition: for $k \leqslant D$, the probability for $x_t < x_{t+k}$ should not depend on t [1].

The advantages of the BPM reside in (a) its simplicity, (b) its robustness, and (c) its invariance with respect to nonlinear monotonous transformations. Also, this method provides an extremely fast computational algorithm. It can be applied to any type of time series (regular, chaotic, noisy, or experimental) [1]. Remark that for the applicability of this approach we need not to assume that the time series under analysis is representative of a low dimensional dynamical system. Of course, the embedding dimension D plays an important role for the evaluation of the appropriate probability distribution, since D determines the number of accessible states, $D!$, and tells us about the necessary length M of the time series needed in order to work with a reliable statistics. In particular, Bandt and Pompe suggest for practical purposes to work with $3 \leqslant D \leqslant 7$. Concerning this last point in all calculations reported here the condition $M \gg D!$ is satisfied [6].

The normalized permutation entropy is just the normalized Shannon entropy associated to the probability distribution $P = \{p(\pi_i), i = 1, \ldots, D!\}$

$$\mathcal{H}_S[P] = S[P]/S_{\max} = \left[-\sum_{i=1}^{D!} p(\pi_i)\ln\bigl(p(\pi_i)\bigr)\right]/S_{\max}, \tag{4}$$

where $S_{\max} = \ln D!$, $(0 \leqslant \mathcal{H}_S \leqslant 1)$—$S$ stands for Shannon entropy.

3. Fractional Brownian motion and fractional Gaussian noise

Fractional Brownian motion (fBm) is the only family of processes which is Gaussian, self-similar,[1] and endowed with stationary increments—see Ref. [19] and references therein. The normalized family of these Gaussian processes, $\{B^H(t), t > 0\}$, is the one with $B^H(0) = 0$ almost surely, i.e., with probability 1, $\mathbb{E}[B^H(t)] = 0$ (zero mean), and covariance given by

$$\mathbb{E}\bigl[B^H(t_1)B^H(t_2)\bigr] = \frac{1}{2}\bigl(t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H}\bigr), \tag{5}$$

for $t_1, t_2 \in \mathbb{R}$. Here $\mathbb{E}[\cdot]$ refers to the average computed with a Gaussian probability density. The power exponent $0 < H < 1$ is commonly known as the Hurst parameter or Hurst exponent. These processes exhibit *memory* for any Hurst parameter except for $H = 1/2$ as one realizes from Eq. (5). The $H = 1/2$-case corresponds to classical Brownian motion and successive motion increments are as likely to have the same sign as the opposite, there is no correlation among them. Thus, Hurst's parameter defines two distinct regions in the interval $(0, 1)$. When $H > 1/2$, consecutive increments tend to have the same sign so that these processes are *persistent*. For $H < 1/2$, on the other hand, consecutive increments are more likely to have opposite signs, thus these processes are *anti-persistent*. Fractional Brownian motions are continuous but non-differentiable processes (in the classical sense). As a non-stationary process, they do not possess a spectrum defined in the usual sense; however, it is possible to define a *generalized power spectrum* of the form

$$\Phi_{B^H}(f) \propto \frac{1}{|f|^{\alpha}}, \tag{6}$$

with the exponent $\alpha = 2H + 1$, $1 < \alpha < 3$.

The fractional Gaussian noise is the process $\{W^H(t), t > 0\}$ obtained from the fBm increments (for discrete time), i.e.,

$$W^H(t) = B^H(t+1) - B^H(t). \tag{7}$$

This is a stationary Gaussian process with mean zero and covariance given by

$$\begin{aligned}\rho(k) &= \mathbb{E}\bigl[W^H(t)W^H(t+k)\bigr] \\ &= \frac{1}{2}\bigl[(k+1)^{2H} - 2k^{2H} + |k-1|^{2H}\bigr], \quad k > 0.\end{aligned} \tag{8}$$

[1] Self-similar stochastic processes are invariant in distribution under suitable scaling of time and space. Formally, a (stochastic) process $X(t)$ is *self-similar* with index H if, for any $c > 0$,

$$X(t) \stackrel{d}{=} c^H X(c^{-1}t),$$

where $\stackrel{d}{=}$ is equality in distribution.