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CPT theorem in a $(5+1)$ Galilean space–time

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Abstract

We extend the 5-dimensional Galilean space–time to a $(5+1)$ Galilean space–time in order to define a parity transformation in a covariant manner. This allows us to discuss the discrete symmetries in the Galilean space–time, which is embedded in the $(5+1)$ Minkowski space–time. We discuss the Dirac-type field, for which we give the 8×8 gamma matrices explicitly. We demonstrate that the CPT theorem holds in the $(5+1)$ Galilean space–time.

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Keywords: CPT invariance; Galilean symmetry; Lorentz symmetry

1. Introduction

Galilean covariance is feasible in an enlarged 5-dimensional manifold, which is embedded in the $(4+1)$ Minkowski space–time. Early works in this direction include Refs. [1–11]; hereafter, we follow more closely the notation in Refs. [6–8]. A 4-dimensional realization of the Clifford algebra in the $(4+1)$ Minkowski space–time requires γ^5 as a fourth "spatial" element of γ_μs. In order to define a γ^5-like matrix, it is necessary to extend the theories to a 6-dimensional manifold [12]. In this Letter, we shall show that it is possible to define, in a covariant manner, the parity matrix, as well as the charge-conjugation and time-reversal matrices in a $(5+1)$ Galilean space–time. Hence we can discuss a CPT theorem based on the transformation properties of the Dirac-type field and state.

In the context of axiomatic field theory, the CPT theorem was established by Wightman [13] and his associates [14,15]. In relativistic quantum field theory, Schwinger proved the spin-statistics connection by symmetrizing (anti-symmetrizing) the kinematical term of the Lagrangian and, hence, the commutation (anticommutation) relations [16,17]. Recently, Puccini and Vucetich axiomatized Schwinger's Lagrangian formulation and proved the CPT theorem by assuming a form of dynamical Lagrangian [18]. Weinberg adopted the causality requirement, which was originally proposed by Pauli [19], in order to prove the connection between spin and statistics as well as the CPT theorem [20].

In Ref. [12], we have developed an 8-dimensional realization of the Clifford algebra in the $(4+1)$ Galilean space–time, in order to define the discrete symmetries, in particular, the parity transformation. This was accomplished by using the dimensional reduction from the $(5+1)$ Minkowski space–time to the $(4+1)$ Minkowski space–time, which, when expressed in terms of light-cone coordinates, corresponds to the $(4+1)$ Galilean space–time.

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In order to avoid using projective representations of the Galilei group, which arise from the existence of a central charge, the $(4+1)$ Galilean space–time was introduced. In the 5-dimensional Galilean theory, an additional coordinate parameter corresponds to a central extension of the group. The usual $(3+1)$ space–time is embedded into the $(4+1)$ Galilean space–time. In odd-dimensional Minkowski space–times, in which the number of spatial coordinates is even, the reflection of spatial manifold has a determinant equal to one, so that it is continuously connected to the identity and can be obtained as a rotation.

Parity refers to a reversal of orientation of the spatial coordinates. Thus we define the parity transformation by the mapping:

$$P' : x^\mu \to x'^\mu = \left(-\mathbf{x}, x^4, x^5\right).$$

However, the existence of this discrete transformation entails the loss of manifest covariance, because the space reflection in the $(4+1)$ Minkowski space–time corresponds to

$$P : x^\mu \to x'^\mu = \left(-\mathbf{x}, x^5, x^4\right).$$

A way to preserve both manifest covariance and the discrete parity operation is to extend the $(4+1)$ Galilean space–time to a $(5+1)$ Galilean space–time. The latter space–time corresponds to a $(5+1)$ Minkowski space–time defined with light-cone coordinates. Motivated by this fact we develop, in this Letter, a 6-dimensional Galilean theory in a covariant manner and prove the CPT theorem.

For the $(5+1)$ Galilean space–time, we use light-cone coordinates x^μ ($\mu = 1, \ldots, 6$), with the metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} 1_{4\times4} & 0_{4\times2} \\ 0_{2\times4} & \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} \end{pmatrix}.$$

Then the coordinate system y^μ ($\mu = 1, \ldots, 5, 0$), defined by

$$\mathbf{y} = \mathbf{x}, \qquad y^4 = x^4, \qquad y^5 = \frac{1}{\sqrt{2}}\left(x^5 - x^6\right), \qquad y^0 = \frac{1}{\sqrt{2}}\left(x^5 + x^6\right), \tag{1}$$

admits the diagonal metric

$$g_{\mu\nu} = \mathrm{diag}(1, 1, 1, 1, 1, -1).$$

This correspondence between the $(5+1)$ Galilean space–time and the $(5+1)$ Minkowski space–time allows us to describe non-relativistic theories in a Lorentz-like covariant manner [6–8], [1–11].

Let Γ^μ and γ^μ denote the 8×8 gamma matrices in the $(5+1)$ Galilean (light-cone coordinates xs) and Minkowski space–times (coordinates ys), respectively. The gamma matrices transform as contravariant vectors in each space–time. Therefore, we have

$$\Gamma^k = \gamma^k = \begin{pmatrix} 0_{4\times4} & \begin{matrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{matrix} \\ \begin{matrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{matrix} & 0_{4\times4} \end{pmatrix}, \quad k = 1, 2, 3, \qquad \Gamma^4 = \gamma^4 = \mathrm{i}\begin{pmatrix} 0_{4\times4} & \begin{matrix} 0 & I \\ I & 0 \end{matrix} \\ \begin{matrix} 0 & -I \\ -I & 0 \end{matrix} & 0_{4\times4} \end{pmatrix},$$

$$\Gamma^5 = \frac{1}{\sqrt{2}}\left(\gamma^5 + \gamma^0\right) = -\sqrt{2}\mathrm{i}\begin{pmatrix} 0_{4\times4} & \begin{matrix} 0 & 0 \\ 0 & I \end{matrix} \\ \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} & 0_{4\times4} \end{pmatrix}, \qquad \Gamma^6 = \frac{1}{\sqrt{2}}\left(-\gamma^5 + \gamma^0\right) = -\sqrt{2}\mathrm{i}\begin{pmatrix} 0_{4\times4} & \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & I \end{matrix} & 0_{4\times4} \end{pmatrix},$$

$$\zeta = \frac{1}{\sqrt{2}}\mathrm{i}\left(\Gamma^5 + \Gamma^6\right) = \mathrm{i}\gamma^0 = \begin{pmatrix} 0_{4\times4} & \begin{matrix} I & 0 \\ 0 & I \end{matrix} \\ \begin{matrix} I & 0 \\ 0 & I \end{matrix} & 0_{4\times4} \end{pmatrix}, \qquad \Gamma^7 = \gamma^7 = \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^0 = \begin{pmatrix} \begin{matrix} I & 0 \\ 0 & I \end{matrix} & 0_{4\times4} \\ 0_{4\times4} & \begin{matrix} -I & 0 \\ 0 & -I \end{matrix} \end{pmatrix}, \tag{2}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that we have inverted γ^4 and γ^5 with respect to Ref. [12].

The canonical conjugate variable of the extended coordinates in the $(5+1)$ Galilean space–time provides a transparent interpretation of the additional parameter s. Indeed, the 6-momentum,

$$\begin{aligned} p_\mu = -\mathrm{i}\partial_\mu &= (-\mathrm{i}\nabla, -\mathrm{i}\partial_4, -\mathrm{i}\partial_t, -\mathrm{i}\partial_s), \\ &= (\mathbf{p}, p_4, -E, -m), \end{aligned}$$

such that $p^5 = -p_6 = m$ and $p^6 = -p_5 = E$, shows that the coordinate s is conjugate to the mass m in the same way that $\mathbf{x}$ is conjugate to the momentum $\mathbf{p}$. Note that we have used $x^5 = t$ (t is the time variable) and $x^6 = s$ in a unit system where $c = 1$.