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Information flow during the quantum-classical transition

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ABSTRACT

We have exhaustively investigated the classical limit of the semi-classical evolution with reference to a well-known model that represents the interaction between matter and a given field. In this Letter we approach this issue by recourse to a new statistical quantifier called the "symbolic transfer entropy" [T. Schreiber, Phys. Rev. Lett. 85 (2000) 461; M. Staniek, K. Lehnertz, Phys. Rev. Lett. 100 (2008) 158101]. We encounter that the quantum-classical transition gets thereby described as the sign reversal of the dominating direction of the information flow between classical and quantal variables. This can be considered as an evidence of the physical useful of this new statistical quantifier.

1. Introduction

The synchronization phenomenon is found in various field of science, engineering, and social behavior [1]. A particularly important example is the coupling between dynamical systems, of general interest since it can be observed in manifold ways. The investigation of the concomitant interactions addresses, as a major aspect, the detection and quantification of the strength and direction (or asymmetry) of couplings. In this vein, much work has been devoted to the problem of assessing directional couplings, including information theoretic approaches. One of these treatments plays a central role here, namely, the one related to the so-called "transfer entropy" (TE) [2]. The TE quantifies the statistical coherence between systems evolving in time and was developed so as to overcome problems with the standard time delayed mutual information, that fails to distinguish between information that is actually exchanged from shared information due to common history and input signals. In the TE these influences are excluded by appropriately conditioning pertinent transition probabilities [2], so that one is able to distinguish effectively driving from responding elements and to detect asymmetry in the interaction of subsystems. It has been shown, via a comparative analysis, that the TE performance is as good as that of other information theoretic measures [3]. Important works relevant to our present effort have been reported, among others, by Marschinski and Kantz [4], Hlaváčková-Schindler et al. [5], Paluš and Vejmelka [6], and Vejmelka and Paluš [7]. For further details see Section 3.

Important progress was also recently made by Staniek and Lehnertz in Ref. [8], where they propose to estimate the transfer entropy by using a particular symbolization approach of the time series under study, the Bandt and Pompe method [9]. This is the only one symbolization technique among those in popular use that takes into account the time causality of the system's dynamics. Then, important details concerning the ordinal structure of the time series can be revealed [10–12]. Staniek and Lehnertz called this statistical quantifier symbolic TE (STE) and demonstrate that it is a robust and computationally fast method to quantify the dominating direction of information flow between time series from coupled systems. The symbolic transfer entropy is an improvement of the transfer entropy for real world applications because it is more robust under the influence of observational noise. Moreover, a resonant-like behavior is found, i.e. the direction of coupling is more easily detected in the presence of noise [8]. It is important to emphasize that for us it is more convenient to use this particular symbolic approach, which does not in any manner mean that it should be considered the superior of the different available techniques.

In this work we apply the STE to the classical limit of quantum mechanics (CLQM) that, contrary to the widespread belief, remains

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an open problem since the problem of the emergence of classical mechanics from quantum mechanics is by no means fully solved. In spite of many results on the $\hbar \to 0$ asymptotics, it is not yet clear how to explain the classical motion of macroscopic bodies within the standard quantum mechanics. In this Letter we shall analyze, *via the STE*, a special case of evolution from quantum to classical behavior [13,14] in the framework of a well-known semi-classical model that represents the interaction of matter with a given field [15,16]. It should be stressed that this model has been intensively studied by some of the authors of this Letter. Different approaches have been considered: I) dynamical analysis with tools like a) Poincaré sections, b) distances between curves (measured with different norms) so as to study the asymptotic classical limit, c) relative number of chaotic curves, etc. [13,14,17,18] and II) statistical analysis by considering quantifiers like the entropy and the statistical complexity. These quantifiers were evaluated in two different ways, a) by employing the wavelet approach (see Refs. [19–21] and references therein) and b) the Bandt and Pompe method (see Refs. [11,22] and references therein). Moreover, the statistical complexity was computed by recourse to two different probability space metrics: Euclidean [19] and Jensen–Shannon divergence [20]. The main results obtained in these previous works are recovered by employing the STE, which enhances the physical significance of this new quantifier.

2. The CLQM for a special semi-classical model

Quite a bit of quantum insight is to be gained from semi-classical perspectives. Several methodologies are available (WKB, Born–Oppenheimer approach, etc.). The model of Refs. [15–17] considers two interacting systems: one of them classical, the other quantal. This makes sense whenever the quantum effects of one of the two systems are negligible in comparison to those of the other one. Examples can be readily found. We can just mention Bloch equations [23], two-level systems interacting with an electromagnetic field within a cavity and Jaynes–Cummings semi-classical model [24–26], collective nuclear motion [27], etc. Thus, we deal here with a special bipartite system that represents the zero-th mode contribution of a strong external field to the production of charged meson pairs [16,17], whose Hamiltonian reads

$$\hat{H} = \frac{1}{2}\left(\frac{\hat{p}^2}{m_q} + \frac{{P_A}^2}{m_{cl}} + m_q\omega^2\hat{x}^2\right), \tag{1}$$

where (i) $\hat{x}$ and $\hat{p}$ are quantum operators, (ii) A and P_A classical canonical conjugate variables, and (iii) $\omega^2 = {\omega_q}^2 + e^2A^2$ is an interaction term that introduces nonlinearity, ω_q being a frequency. The quantities m_q and m_{cl} are masses, corresponding to the quantum and classical systems, respectively. As shown in Refs. [13,14], in dealing with Eq. (1) one faces an autonomous system of nonlinear coupled equations

$$\begin{aligned}
&\frac{d\langle\hat{x}^2\rangle}{dt} = \frac{\langle\hat{L}\rangle}{m_q}, \qquad \frac{d\langle\hat{p}^2\rangle}{dt} = -m_q\omega^2\langle\hat{L}\rangle, \\
&\frac{d\langle\hat{L}\rangle}{dt} = 2\left(\frac{\langle\hat{p}^2\rangle}{m_q} - m_q\omega^2\langle\hat{x}^2\rangle\right), \\
&\frac{dA}{dt} = \frac{P_A}{m_{cl}}, \qquad \frac{dP_A}{dt} = -e^2 m_q A\langle\hat{x}^2\rangle,
\end{aligned} \tag{2}$$

where $\hat{L} = \hat{x}\hat{p} + \hat{p}\hat{x}$. The system of equations (2) follows immediately from Ehrenfest's relations [13]. To study the classical limit we also need to consider the classical counterpart of the Hamiltonian given by Eq. (1)

$$H = \frac{1}{2}\left(\frac{p^2}{m_q} + \frac{{P_A}^2}{m_{cl}} + m_q\omega^2 x^2\right), \tag{3}$$

where all the variables are classical. Recourse to Hamilton's equations allows one to find the classical version of Eqs. (2); see Ref. [14] for further details. These equations are identical in form to Eqs. (2) after suitable replacement of quantum mean values by classical variables, i.e., $\langle\hat{x}^2\rangle \Rightarrow x^2$, $\langle\hat{p}^2\rangle \Rightarrow p^2$ and $\langle\hat{L}\rangle \Rightarrow L = 2xp$. The classical limit is obtained by letting the "relative energy"

$$E_r = \frac{|E|}{I^{1/2}\omega_q} \to \infty, \tag{4}$$

where E is the total energy of the system and I is an invariant of the motion described by the system of equations previously introduced (Eqs. (2)), related to the Uncertainty Principle

$$I = \langle\hat{x}^2\rangle\langle\hat{p}^2\rangle - \frac{\langle\hat{L}\rangle^2}{4} \geqslant \frac{\hbar^2}{4}. \tag{5}$$

A classical computation of I yields $I = x^2p^2 - L^2/4 \equiv 0$. A measure of the degree of convergence between classical and quantum results in the limit of Eq. (4) is given by the norm $\mathcal{N}$ of the vector $\Delta u = u - u_{cl}$ [14]

$$\mathcal{N}_{\Delta u} = |u - u_{cl}|, \tag{6}$$

where the three components vector $u = (\langle\hat{x}^2\rangle, \langle\hat{p}^2\rangle, \langle\hat{L}\rangle)$ is the "quantum" part of the solution of the system defined by Eqs. (2) and $u_{cl} = (x^2, p^2, L)$ its classical counterpart.

A detailed study of this model was performed in Refs. [13,14,17]. The main results of these references, pertinent for our discussion, can be succinctly detailed as follows: in plotting diverse dynamical quantities as a function of E_r (as it grows from unity to ∞), one finds *an abrupt change in the system's dynamics for a special value of* E_r, *to be denoted by* ${E_r}^{cl} = 21.55264$. From this value onwards, the pertinent dynamics starts converging to the classical one. It is thus possible to assert that ${E_r}^{cl}$ provides us with an *indicator* of the quantum-classical "border". The zone

$$E_r < {E_r}^{cl} \tag{7}$$

corresponds to the semi-quantal regime investigated in Ref. [17]. This regime, in turn, is characterized by *two* different sub-zones [14]. One of them is an almost purely quantal one, in which the microscopic quantal oscillator is just slightly perturbed by the classical one, and the other section exhibits a transitional nature. The border between these two sub-zones can be well characterized by a relative energy value ${E_r}^{\mathcal{P}} = 3.3282$. A significant feature of this point resides in the fact that, for $E_r \geqslant {E_r}^{\mathcal{P}}$, *chaos is always found*. The relative number of chaotic orbits (with respect to the total number of orbits) grows with E_r and tends to unity for $E_r \to \infty$ [14,17].

Thus, as E_r grows from $E_r = 1$ (the "pure quantum instance") to $E_r \to \infty$ (the classical situation), a significant series of *morphology changes* is detected, specially in the transition zone (${E_r}^{\mathcal{P}} \leqslant E_r < {E_r}^{cl}$). The concomitant orbits exhibit features that are not easily describable in terms of Eq. (6), which is a *global* measure of the degree of convergence in amplitude (of the signal). What one needs instead is *a statistical type of characterization* involving the notions of entropy and statistical complexity. As it was mentioned at the Introduction these quantifiers were evaluated in various ways as, for instance, by employing the wavelet approach (see Refs. [19–21] and references therein) or the Bandt and Pompe method (see Refs. [11,22] and references therein). These two statistical quantifiers are able to adequately identify the properties of the three zones that cover the quantum-classical evolution (as E_r varies).

Additionally, statistical analysis forces our attention towards another relevant, orbit-dependent E_r value within the transition zone, namely E_r^M, located in the E_r interval [6, 8], where the sta-