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Spin description in the star product and path integral formalisms

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Abstract

Spin can be described in the star product formalism by extending the bosonic Moyal product in the fermionic sector. One can then establish the relation to other approaches that describe spin with fermionic variables. The fermionic star product formalism and the fermionic path integral formalism are related in analogy to their bosonic counterparts.

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1. Introduction

The star product formalism as it was established by Bayen et al. in [1] gives an alternative description of quantum mechanics on the phase space [2]. One might then wonder if it is also possible to describe spin in the star product formalism. Berezin and Marinov showed in [3] how the spin can be described in the context of pseudoclassical mechanics. So it appears natural to apply the program of deformation quantization to pseudoclassical mechanics in order to obtain a description of spin with a fermionic star product [4]. With the fermionic star product it is then possible to find in analogy to the bosonic Moyal star product formalism spin Wigner functions as eigenfunctions of a fermionic star eigenvalue equation and a spin star exponential that describes the time development.

A fermionic star product already appeared in [1], whereas a systematic account to a fermionic star product formalism was given in [5]. With the fermionic star product the Grassmann algebra is deformed into a Clifford algebra, so that such a Clifford star product can be interpreted as the geometric product of geometric algebra in a superanalytic formulation [6]. Geometric algebra goes back to early ideas of Hamilton, Grassmann, and Clifford and was developed into a full formalism by Hestenes in [7] and [8]. From then on it was applied to a wide range of physical problems and in particular it was used for the description of spin (for a thorough discussion of geometric algebra see [9]). The superanalytic formulation of geometric algebra and its relation to pseudoclassical mechanics was established in [10] so that—in conjunction with the fermionic Clifford star product—a natural geometric interpretation of the fermionic star product formalism is obtained. Furthermore, it now appears natural to combine the fermionic and the bosonic Moyal star product formalism (see for example [11]). This leads then to a noncommutative version of geometric algebra, where the noncommutativity leads to the natural appearance of a spin term [6] and for geometric algebra on the phase space to a natural appearance of supersymmetric quantum mechanics [12]. Another possibility to connect noncommutativity by star products and spin was discussed in [13].

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In the next section we will first give a short overview of the formal structures in the Moyal star product formalism so that one can compare this to the fermionic case. In Section 3 we will then give a short introduction to geometric algebra in its superanalytic formulation with a Clifford star product. Thereafter, we show how the spin arises naturally from noncommutative geometric algebra and how it can also be formulated in two isomorphic Clifford algebras. Furthermore, we establish the connection of the spin description by spin projectors or spin Wigner functions to the spinor description in [9]. In Sections 6 and 7 we then describe how the spin can be formulated with a fermionic operator and a fermionic path integral formalism. For both approaches the relation to the fermionic star product formalism is established. In particular we show that the relation of the fermionic path integral and the fermionic star product formalism is analogous to the bosonic one that was found in [14,15].

2. The Moyal star product formalism

On the $2d$-dimensional flat phase space with coordinates (q^i, p^i) the Moyal product of two phase space functions $f(q,p)$ and $g(q,p)$ is defined as

$$f *_M g = f \exp\left[\frac{\mathrm{i}\hbar}{2}\sum_{i=1}^{d}\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p^i} - \frac{\overleftarrow{\partial}}{\partial p^i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)\right] g. \tag{2.1}$$

It can also be written in integral form, which in a two-dimensional phase space reads [16]

$$f *_M g = \frac{1}{\hbar^2\pi^2}\int dq'\,dq''\,dp'\,dp''\, f(q',p')g(q'',p'')\exp\left(\frac{2}{\mathrm{i}\hbar}\big(p(q'-q'') + p'(q''-q) + p''(q-q')\big)\right). \tag{2.2}$$

The star product replaces the conventional product between functions on the phase space and it is so constructed that the star anticommutator corresponds to the Poisson bracket:

$$\lim_{\hbar\to 0}\frac{1}{\mathrm{i}\hbar}[f,g]_{*_M} = \lim_{\hbar\to 0}\frac{1}{\mathrm{i}\hbar}(f *_M g - g *_M f) = \{f,g\}_{\mathrm{PB}}. \tag{2.3}$$

This relation is the principle of correspondence. The states of a system with Hamilton function $H(q,p)$ are described by Wigner functions $\pi_n^{(M)}(q,p)$. The Wigner functions and the corresponding energy levels E_n can be calculated with the help of the star exponential $\mathrm{Exp}_M(Ht)(q,p)$, which is defined as

$$\mathrm{Exp}_M(Ht) = e_{*_M}^{-\frac{\mathrm{i}t}{\hbar}H} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\mathrm{i}t}{\hbar}\right)^n H^{n*_M} = \sum_{n=0}^{\infty}\pi_n^{(M)}e^{-\mathrm{i}E_n t/\hbar}, \tag{2.4}$$

where $H^{n*_M} = H *_M \cdots *_M H$ is the n-fold star product of H. The star exponential fulfills the analogue of the time dependent Schrödinger equation

$$\mathrm{i}\hbar\frac{d}{dt}\mathrm{Exp}_M(Ht) = H *_M \mathrm{Exp}_M(Ht) \tag{2.5}$$

and describes the time development of the Wigner function $\pi_n^{(M)}(q,p)$ [2]:

$$\pi_n^{(M)}(t) = \overline{\mathrm{Exp}_M(Ht)} *_M \pi_n^{(M)} *_M \mathrm{Exp}_M(Ht). \tag{2.6}$$

The connection of the star exponential and the path integral was established in [14,15]. It was shown that the path integral is the Fourier transform of the star exponential, i.e.

$$\int\frac{Dq\,Dp}{2\pi\hbar}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f}(p\dot{q}-H)\,dt\right] = \int\frac{dp}{2\pi\hbar}\exp\left[\frac{\mathrm{i}}{\hbar}p(q_f-q_i)\right]\mathrm{Exp}_M(Ht)\big((q_f+q_i)/2,p\big). \tag{2.7}$$

Putting now (2.4) into (2.5) gives the $*_M$-eigenvalue equation

$$H *_M \pi_n^{(M)} = E_n\pi_n^{(M)} \tag{2.8}$$

and for $t=0$ these equations lead to the spectral decomposition of the Hamilton function:

$$H = \sum_{n=0}^{\infty}E_n\pi_n^{(M)}. \tag{2.9}$$

Substituting this expression in (2.8) gives

$$\pi_n^{(M)} *_M \pi_m^{(M)} = \delta_{mn}\pi_n^{(M)}, \tag{2.10}$$

which together with the completeness relation $\sum_{n=0}^{\infty}\pi_n^{(M)} = 1$, that follows from (2.4) for $t=0$, means that the Wigner functions are projectors.