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Non-Markovian random unitary qubit dynamics

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ABSTRACT

We compare two approaches to non-Markovian quantum evolution: one based on the concept of divisible maps and the other one based on distinguishability of quantum states. The former concept is fully characterized in terms of local generator whereas it is in general not true for the latter one. A simple example of random unitary dynamics of a qubit shows the intricate difference between those approaches. Moreover, in this case both approaches are fully characterized in terms of local decoherence rates.

1. Introduction

Quantum dynamics is represented by the dynamical map, that is, a family of completely positive and trace preserving maps Λ_t ($t \geqslant 0$) such that $\Lambda_0 = \mathbb{1}$. If ρ is an initial state of the system then $\rho_t = \Lambda_t(\rho)$ defines evolution of ρ. This description provides generalization of unitary evolution $\Lambda_t(\rho) = U_t \rho U_t^\dagger$ with unitary $U_t = e^{-iHt}$, where H represents the Hamiltonian of the system. Any departure from unitary evolution signals the nontrivial interaction of the system with an environment which is responsible for decoherence and dissipation processes [1] in open quantum system. The traditional approach to the dynamics of such systems consists in applying a suitable Born–Markov approximation leading to the celebrated quantum Markov semigroup [2–4] which neglects all memory effects. Recent theoretical activity and technological progress show the importance of more refine approach based on non-Markovian evolution. Non-Markovian quantum dynamics becomes in recent years very active field of both theoretical and experimental research (see e.g. recent papers [5–29] and references therein).

Throughout this Letter we call quantum evolution Markovian if the corresponding dynamical map Λ_t is divisible [16]. Let us recall that Λ_t is divisible if $\Lambda_t = V_{t,s}\Lambda_s$ and $V_{t,s}$ is completely positive and trace preserving for all $t \geqslant s$, that is, it gives rise to 2-parameter family of legitimate propagators. The essential property of $V_{t,s}$ is the following (inhomogeneous) composition law $V_{t,s}V_{s,u} = V_{t,u}$ for all $t \geqslant s \geqslant u$. It should be stressed that Markovian dynamics (divisible map) is entirely characterized by the properties of the local in time generators L_t. It means that if Λ_t satisfies the time-local Master Equation $\dot{\Lambda}_t = L_t \Lambda_t$, then Λ_t corresponds to Markovian dynamics if and only if the local generator time-dependent L_t has the standard form [2,3] for all $t \geqslant 0$, that is,

$$L_t\rho = -i\big[H(t), \rho\big] + \sum_\alpha \left(V_\alpha(t)\rho V_\alpha^\dagger(t) - \frac{1}{2}\big\{V_\alpha^\dagger(t)V_\alpha(t), \rho\big\}\right),$$

with time-dependent Hamiltonian $H(t)$ and noise operators $V_\alpha(t)$. A different approach to Markovianity was recently proposed by Breuer et al. [17]: authors of [17] define non-Markovian dynamics as a time evolution for the open system characterized by a temporary flow of information from the environment back into the system and manifests itself as an increase in the distinguishability of pairs of evolving quantum states:

$$\sigma(\rho_1, \rho_2; t) = \frac{1}{2}\frac{d}{dt}\big\|\Lambda_t(\rho_1 - \rho_2)\big\|_1, \tag{1}$$

where $\|A\|_1 = \mathrm{Tr}\sqrt{A^\dagger A}$ denotes the trace norm. According to [17] the dynamics Λ_t is Markovian iff $\sigma(\rho_1, \rho_2; t) \leqslant 0$ for all pairs of states ρ_1, ρ_2 and $t \geqslant 0$. Contrary to divisible map this approach does not correspond to any composition law and is not characterized by the properties of local generator. It turns out that condition $\sigma(\rho_1, \rho_2; t) \leqslant 0$ is less restrictive than requirement of complete positivity for $V_{t,s}$ and one can construct Λ_t which is non-Markovian (not divisible) but still gives rise to the negative flow of information (see [13–15]).

In this Letter we provide a simple example of qubit dynamics for which the condition $\sigma(\rho_1, \rho_2; t) \leqslant 0$ is fully characterized by local generator. We stress that it is the first example of a legitimate qubit dynamical map for which (i) the divisibility and the condition for negative information flow define two different classes of evolutions, and (ii) the information flow is fully controlled by local

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decoherence rates. We discuss possible generalization for random unitary dynamics of N-level quantum systems.

2. Random unitary dynamics: divisibility vs. information flow

Consider the following random unitary dynamical map

$$\Lambda_t\rho = \sum_{\alpha=0}^{3} p_\alpha(t)\sigma_\alpha\rho\sigma_\alpha, \tag{2}$$

where $\sigma_0 = \mathbb{I}$, and σ_1, σ_2, σ_3 are Pauli matrices, and $p_\alpha(t)$ is time-dependent probability distribution such that $p_0(0) = 1$ which guarantee that $\Lambda_0 = \mathbb{1}$. This map was recently investigated in [28]. It is clear that one may consider (2) as a family of Pauli channels. To answer the question whether Λ_t is Markovian one has to analyze the corresponding local generator. To find $L_t = \dot{\Lambda}_t\Lambda_t^{-1}$ one has to compute the inverse map Λ_t^{-1}. Let us observe that

$$\Lambda_t(\sigma_\alpha) = \lambda_\alpha(t)\sigma_\alpha, \tag{3}$$

where the time-dependent eigenvalues are given by

$$\lambda_\alpha(t) = \sum_{\beta=0}^{3} H_{\alpha\beta}p_\beta(t), \tag{4}$$

with $H_{\alpha\beta}$ being the Hadamard matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{5}$$

Note that $\lambda_0(t) = 1$ and $|\lambda_k(t)| \leqslant 1$ for $k = 1, 2, 3$. It is therefore clear that

$$L_t(\sigma_\alpha) = \mu_\alpha(t)\sigma_\alpha, \tag{6}$$

where

$$\mu_\alpha(t) = \frac{\dot{\lambda}_\alpha(t)}{\lambda_\alpha(t)}, \tag{7}$$

and hence in particular $\mu_0(t) = 0$ due to $\lambda_0(t) = 1$. Introducing

$$\gamma_\alpha(t) = \frac{1}{4}\sum_{\beta=0}^{3} H_{\alpha\beta}\mu_\beta(t), \tag{8}$$

the local generator L_t is given by

$$L_t(\rho) = \sum_{\alpha=0}^{3} \gamma_\alpha(t)\sigma_\alpha\rho\sigma_\alpha. \tag{9}$$

Finally, observing that

$$\sum_{\alpha=0}^{3} \gamma_\alpha(t) = \frac{1}{4}\sum_{\alpha,\beta=0}^{3} H_{\alpha\beta}\mu_\beta(t) = \mu_0(t) = 0,$$

one arrives at the standard form

$$L_t(\rho) = \sum_{k=1}^{3} \gamma_k(t)(\sigma_k\rho\sigma_k - \rho). \tag{10}$$

Hence

Proposition 1. *The random unitary dynamics (2) is Markovian (i.e. Λ_t is divisible) iff*

$$\gamma_1(t) \geqslant 0, \qquad \gamma_2(t) \geqslant 0, \qquad \gamma_3(t) \geqslant 0, \tag{11}$$

for all $t \geqslant 0$.

Note, that (11) yields highly nontrivial conditions for probability distribution $p_\alpha(t)$:

$$\sum_{\beta=0}^{3} H_{k\beta}\left\{\frac{\sum_{\nu=0}^{3} H_{\beta\nu}\dot{p}_\nu(t)}{\sum_{\sigma=0}^{3} H_{\beta\sigma}p_\sigma(t)}\right\} \geqslant 0, \tag{12}$$

for $k = 1, 2, 3$ and $t \geqslant 0$. The inverse relations yield

$$p_0(t) = \frac{1}{4}\big[1 + A_{12}(t) + A_{13}(t) + A_{23}(t)\big],$$
$$p_1(t) = \frac{1}{4}\big[1 - A_{12}(t) - A_{13}(t) + A_{23}(t)\big],$$
$$p_2(t) = \frac{1}{4}\big[1 - A_{12}(t) + A_{13}(t) - A_{23}(t)\big],$$
$$p_3(t) = \frac{1}{4}\big[1 + A_{12}(t) - A_{13}(t) - A_{23}(t)\big],$$

where $A_{ij}(t) = e^{-2[\Gamma_i(t)+\Gamma_j(t)]}$ and we introduced

$$\Gamma_k(t) = \int_0^t \gamma_k(\tau)\,d\tau.$$

Let us recall [26] that if Λ_t is Markovian then

$$\frac{d}{dt}\big\|\Lambda_t(X)\big\|_1 \leqslant 0, \tag{13}$$

for all Hermitian X. Taking $X = \sigma_k$ one obtains

$$\frac{d}{dt}\big\|\Lambda_t(\sigma_k)\big\|_1 = \frac{d}{dt}\big|\lambda_k(t)\big|\|\sigma_k\|_1 \leqslant 0, \tag{14}$$

and hence Markovianity of random unitary evolution (2) implies

$$\frac{d}{dt}\big|\lambda_k(t)\big| \leqslant 0, \tag{15}$$

for $k = 1, 2, 3$. One easily finds the following relations

$$\lambda_1(t) = e^{-2[\Gamma_2(t)+\Gamma_3(t)]},$$
$$\lambda_2(t) = e^{-2[\Gamma_1(t)+\Gamma_3(t)]},$$
$$\lambda_3(t) = e^{-2[\Gamma_1(t)+\Gamma_2(t)]},$$

and hence (15) implies

$$\begin{aligned} &\gamma_1(t) + \gamma_2(t) \geqslant 0, \\ &\gamma_1(t) + \gamma_3(t) \geqslant 0, \\ &\gamma_2(t) + \gamma_3(t) \geqslant 0, \end{aligned} \tag{16}$$

for all $t \geqslant 0$. We stress that the above conditions are only necessary but not sufficient for Markovianity: (11) imply (16) but the converse is of course not true.

Now comes our main result:

Proposition 2. *The random unitary dynamics (2) satisfies $\sigma(\rho_1, \rho_2; t) \leqslant 0$ if and only if conditions (16) are satisfied.*

Proof. Let us consider $\|\Delta_t\|_1$, where $\Delta_t = \Lambda_t(\Delta)$ and $\Delta = \rho_1 - \rho_2$. Since Δ is traceless one has $\Delta = \sum_{k=1}^{3} x_k\sigma_k$ with real x_k. Note that $\Delta^2 = (\sum_{k=1}^{3} x_k^2)\mathbb{I}$. Now, $\Lambda_t(\Delta) = \sum_{k=1}^{3}\lambda_k(t)x_k\sigma_k$ is traceless as well, and hence

$$\big\|\Lambda_t(\Delta)\big\|_1 = \mathrm{Tr}\sqrt{\big[\Lambda_t(\Delta)\big]^2} = \mathrm{Tr}\big[\xi(t)\mathbb{I}\big] = 2\xi(t), \tag{17}$$